Chapter 6. Rigid Body Motion

This chapter discusses the motion of rigid bodies, with a focus on their rotation. Some byproduct results of this analysis will enable us to discuss, in the end of the chapter, the description of motion of point particles in non-inertial reference frames.

6.1. Angular velocity vector

Our study of 1D waves in the past chapter has prepared us to for a discussion of 3D systems of particles. We will start it with a (relatively :-) simple limit when the changes of distances $r_{kk'} \equiv |\mathbf{r}_k - \mathbf{r}_{k'}|$ between particles of the system are negligibly small. Such an abstraction is called the (*absolutely*) *rigid body*, and is a reasonable approximation in many practical problems, including the motion of solids. In this model we neglect *deformations* - that will be the subject of the next two chapters.

The rigid body approximation reduces the number of degrees of freedom of the system from $3N$ to just 6 - for example, 3 Cartesian coordinates of one point (say, O), and 3 angles of the system rotation about 3 mutually perpendicular axes passing through this point. (An alternative way to arrive at the same number 6 is to consider 3 points of the body, which uniquely define its position. If movable independently, the points would have 9 degrees of freedom, but since 3 distances $r_{kk'}$ between them are now fixed, the resulting 3 constraints reduce the number of degrees of freedom to 6.)

Let us show that an arbitrary elementary displacement of such a rigid body may be always considered as a sum of a translational motion and a rotation. Consider a "moving" reference frame, firmly bound to the body, and an arbitrary vector **A** – see Fig. 1.

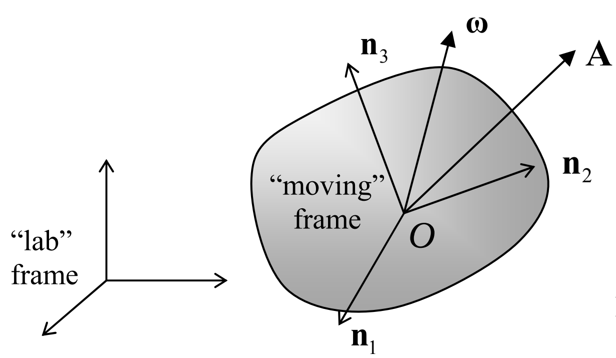

Fig. 6.1. Deriving Eq. (8).

The vector may be represented by its Cartesian components A_j in that reference frame:

$$\mathbf{A} = \sum_{j=1}^{3} A_j \mathbf{n}_j \,. \qquad (6.1)$$

Let us calculate its time derivative in an arbitrary, possibly different ("lab") frame, taking into account that if the body rotates relative to this frame, then the directions of the unit vectors $\mathbf{n}_j$ change in time. Hence, we have to differentiate both operands in each product contributing to sum (1):

$$\left.\frac{d\mathbf{A}}{dt}\right|_{\text{in lab}} = \sum_{j=1}^{3} \frac{dA_j}{dt}\mathbf{n}_j + \sum_{j=1}^{3} A_j \frac{d\mathbf{n}_j}{dt} \,. \qquad (6.2)$$

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In this expression, the first sum evidently describes the change of vector **A** as observed from the moving frame. Each of the infinitesimal vectors $d\mathbf{n}_j$ participating in the second sum may be presented by its Cartesian components in the moving frame:

$$d\mathbf{n}_j = \sum_{j'=1}^{3} d\varphi_{jj'}\mathbf{n}_{j'} . \tag{6.3}$$

In order to find more about the set of scalar coefficients $d\varphi_{jj'}$, let us scalar-multiply each part of this relation by an arbitrary unit vector $\mathbf{n}_{j''}$, and take into account the evident orthogonality condition:

$$\mathbf{n}_{j'} \cdot \mathbf{n}_{j''} = \delta_{j'j''} . \tag{6.4}$$

As a result, we get

$$d\varphi_{jj''} = d\mathbf{n}_j \cdot \mathbf{n}_{j''} . \tag{6.5}$$

Now let us use Eq. (5) to calculate the first differential of Eq. (4):

$$d\mathbf{n}_{j'} \cdot \mathbf{n}_{j''} + \mathbf{n}_{j'} \cdot d\mathbf{n}_{j''} = d\varphi_{j'j''} + d\varphi_{j''j'} = 0; \quad \text{in particular,} \quad 2d\mathbf{n}_j \cdot \mathbf{n}_j = 2d\varphi_{jj} = 0 . \tag{6.6}$$

These relations, valid for any choice of indices j, j', and j'' of the set {1, 2, 3}, mean that the matrix of elements $d\varphi_{jj'}$ is antisymmetric; in other words, there are not 9, but just 3 independent coefficients $d\varphi_{jj'}$, all with $j \neq j'$. Hence it is natural to renumber them in a simpler way: $d\varphi_{jj'} = - d\varphi_{j'j} \equiv d\varphi_{j''}$, where indices j, j', and j'' follow in a "correct" order - either {1,2,3}, or {2,3,1}, or {3,1,2}. Now it is easy to check (say, just by a component-by-component comparison) that in this new notation, Eq. (3) may be presented just as a vector product:

$$d\mathbf{n}_j = d\boldsymbol{\varphi} \times \mathbf{n}_j , \tag{6.7}$$

Elementary rotation

where $d\boldsymbol{\varphi}$ is the infinitesimal vector defined by its Cartesian components $d\varphi_j$ (in the moving frame).

Relation (7) is the basis of all rotation kinematics. Using it, Eq. (2) may be rewritten as

$$\left.\frac{d\mathbf{A}}{dt}\right|_{\text{in lab}} = \left.\frac{d\mathbf{A}}{dt}\right|_{\text{in mov}} + \sum_{j=1}^{3} A_j \frac{d\boldsymbol{\varphi}}{dt} \times \mathbf{n}_j = \left.\frac{d\mathbf{A}}{dt}\right|_{\text{in mov}} + \boldsymbol{\omega} \times \mathbf{A}, \quad \text{where } \boldsymbol{\omega} \equiv \frac{d\boldsymbol{\varphi}}{dt}. \tag{6.8}$$

Vector's evolution in time

In order to interpret the physical sense of vector $\boldsymbol{\omega}$, let us apply Eq. (8) to the particular case when **A** is the radius-vector **r** of a point of the body, and the lab frame is selected in a special way: its origin moves with the same velocity as that of the moving frame in the particular instant under consideration. In this case the first term in the right-hand part of Eq. (8) is zero, and we get

$$\left.\frac{d\mathbf{r}}{dt}\right|_{\text{in special lab frame}} = \boldsymbol{\omega} \times \mathbf{r} , \tag{6.9}$$

were vector **r** is the same in both frames. According to the vector product definition, the particle velocity described by this formula has a direction perpendicular to vectors $\boldsymbol{\omega}$ and **r** (Fig. 2), and magnitude $\omega r \sin\theta$. As Fig. 2 shows, this expression may be rewritten as $\omega\rho$, where $\rho = r\sin\theta$ is the distance from the line that is parallel to vector $\boldsymbol{\omega}$ and passes through point O. This is of course just the pure *rotation* about that line (called the *instantaneous axis of rotation*), with angular velocity ω. Since, according to Eqs. (3) and (8), the *angular velocity vector* $\boldsymbol{\omega}$ is defined by the time evolution of the moving frame alone, it is the same for all points **r**, i.e. for the rigid body as a whole. Note that nothing in our calculations forbids not only the magnitude but also the direction of vector $\boldsymbol{\omega}$, and thus of the instantaneous axis of rotation, to change in time (and in many cases it does); hence the name.

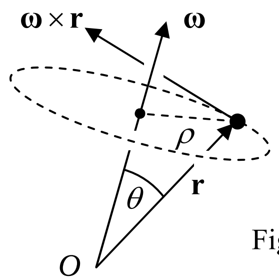

Fig. 6.2. Instantaneous axis of rotation.

Now let us generalize our result a step further, considering two laboratory reference frames that do not rotate versus each other: one arbitrary, and another one selected in the special way described above, so that for it Eq. (9) is valid in it. Since their relative motion of these two reference frames is purely translational, we can use the simple velocity addition rule given by Eq. (1.8) to write

Body point's velocity

$$\mathbf{v}\big|_{\text{in lab}} = \mathbf{v}_O\big|_{\text{in lab}} + \mathbf{v}\big|_{\text{in special lab frame}} = \mathbf{v}_O\big|_{\text{in lab}} + \boldsymbol{\omega}\times\mathbf{r}, \tag{6.10}$$

where $\mathbf{r}$ is the radius-vector of a point is measured in the body-bound ("moving") frame O.

6.2. Inertia tensor

Since the dynamics of each point of a rigid body is strongly constrained by conditions $r_{kk'}$ = const, this is one of the most important fields of application of the Lagrangian formalism that was discussed in Chapter 2. The first thing we need to know for using this approach is the kinetic energy of the body in an inertial reference frame. It is just the sum of kinetic energies of all its points, so that we can use Eq. (10) to write:[1]

$$T = \sum \frac{m}{2}\mathbf{v}^2 = \sum \frac{m}{2}(\mathbf{v}_O + \boldsymbol{\omega}\times\mathbf{r})^2 = \sum \frac{m}{2}v_O^2 + \sum m\mathbf{v}_O\cdot(\boldsymbol{\omega}\times\mathbf{r}) + \sum \frac{m}{2}(\boldsymbol{\omega}\times\mathbf{r})^2. \tag{6.11}$$

Let us apply to the right-hand part of Eq. (11) two general vector analysis formulas, listed in the Math Appendix: the operand rotation rule MA Eq. (7.6) to the second term, and MA Eq. (7.7b) to the third term. The result is

$$T = \sum \frac{m}{2}v_O^2 + \sum m\mathbf{r}\cdot(\mathbf{v}_O\times\boldsymbol{\omega}) + \sum \frac{m}{2}\left[\omega^2 r^2 - (\boldsymbol{\omega}\cdot\mathbf{r})^2\right]. \tag{6.12}$$

This expression may be further simplified by making a specific choice of point O (from the radius-vectors $\mathbf{r}$ of all particles are measured), namely if we use for this point the center of mass of the body. As was already mentioned in Sec. 3.4, radius-vector $\mathbf{R}$ of this point is defined as

$$M\mathbf{R} \equiv \sum m\mathbf{r}, \quad M \equiv \sum m, \tag{6.13}$$

[1] Actually, all symbols for particle masses, coordinates and velocities should carry the particle index, say k, over which the summation is carried out. However, for the sake of notation simplicity, this index is just implied.

where M is just the total mass of the body. In the reference frame centered as that point, $\mathbf{R} = 0$, so that in that frame the second sum in Eq. (12) vanishes, so that the kinetic energy is a sum of two terms:

$$T = T_{\text{tran}} + T_{\text{rot}}, \quad T_{\text{tran}} \equiv \frac{M}{2}V^2, \quad T_{\text{rot}} \equiv \sum \frac{m}{2}\left[\omega^2 r^2 - (\boldsymbol{\omega}\cdot\mathbf{r})^2\right], \tag{6.14}$$

where $\mathbf{V} \equiv d\mathbf{R}/dt$ is the center-of-mass velocity in our inertial reference frame, and all particle positions $\mathbf{r}$ have to be measured in the center-of-mass frame. Since the angular velocity vector $\boldsymbol{\omega}$ is common for all points of a rigid body, it is more convenient to rewrite the rotational energy in a form in which the summation over the components of this vector is clearly separated from the summation over the points of the body:

Kinetic energy of rotation

$$T_{\text{rot}} = \frac{1}{2}\sum_{j,j'=1}^{3} I_{jj'}\omega_j\omega_{j'}, \tag{6.15}$$

where the 3×3 matrix with elements

Inertia tensor

$$I_{jj'} \equiv \sum m\left(r^2\delta_{jj'} - r_j r_{j'}\right) \tag{6.16}$$

is called the *inertia tensor* of the body.

Actually, the term "tensor" for the matrix has to be justified, because in physics this name implies a certain reference-frame-independent notion, so that its elements have to obey certain rules at the transfer between reference frames. In order to show that the inertia tensor deserves its title, let us calculate another key quantity, the total angular momentum $\mathbf{L}$ of the same body.[2] Summing up the angular momenta of each particle, defined by Eq. (1.31), and using Eq. (10) again, in our inertial reference frame we get

$$\mathbf{L} \equiv \sum \mathbf{r}\times\mathbf{p} = \sum m\mathbf{r}\times\mathbf{v} = \sum m\mathbf{r}\times(\mathbf{v}_O + \boldsymbol{\omega}\times\mathbf{r}) = \sum m\mathbf{r}\times\mathbf{v}_O + \sum m\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r}). \tag{6.17}$$

We see that the momentum may be presented as a sum of two terms. The first one,

$$\mathbf{L}_O \equiv \sum m\mathbf{r}\times\mathbf{v}_O = M\mathbf{R}\times\mathbf{v}_O, \tag{6.18}$$

describes possible rotation of the center of mass about the inertial frame origin. This term evidently vanishes if the moving reference frame's origin O is positioned at the center of mass. In this case we are left with only the second term, which describes the rotation of the body about its center of mass:

$$\mathbf{L} = \mathbf{L}_{\text{rot}} \equiv \sum m\mathbf{r}\times(\boldsymbol{\omega}\times\mathbf{r}). \tag{6.19}$$

Using one more vector algebra formula, the "bac minis cab" rule,[3] we may rewrite this expression as

$$\mathbf{L} = \sum m\left[\boldsymbol{\omega}r^2 - \mathbf{r}(\mathbf{r}\cdot\boldsymbol{\omega})\right]. \tag{6.20}$$

Let us spell out an arbitrary Cartesian component of this vector:

[2] Hopefully, there is a little chance of confusion between the angular momentum $\mathbf{L}$ (a vector) and its Cartesian components L_j (scalars with an index) on one hand, and the Lagrange function L (a scalar without an index) on the other hand.

[3] See, e.g., MA Eq. (7.5).

$$L_j = \sum m\left[\omega_j r^2 - r_j \sum_{j'=1}^{3} r_{j'}\omega_{j'}\right] = \sum m \sum_{j'=1}^{3} \omega_{j'}\left(r^2\delta_{jj'} - r_j r_{j'}\right). \tag{6.21}$$

Changing the order of summations, and comparing the result with Eq. (16), we see that the angular momentum may be conveniently expressed via the same matrix elements $I_{jj'}$ as the rotational kinetic energy:

Angular momentum

$$L_j = \sum_{j'=1}^{3} I_{jj'}\omega_{j'} . \tag{6.22}$$

Since **L** and **ω** are both legitimate vectors (meaning that they describe physical vectors independent on the reference frame choice), their connection, the matrix of elements $I_{jj'}$, is a legitimate tensor. This fact, and the symmetry of the tensor ($I_{jj'} = I_{j'j}$), which is evident from its definition (16), allow the tensor to be further simplified. In particular, mathematics tells us that by a certain choice of the axis orientation, any symmetric tensor may be reduced to a diagonal form

$$I_{jj'} = I_j \delta_{jj'}, \tag{6.23}$$

where, in our case

Principal moments of inertia

$$I_j = \sum m\left(r^2 - r_j^2\right) = \sum m\left(r_{j'}^2 + r_{j''}^2\right) = \sum m\rho_j^2 , \tag{6.24}$$

ρ_j being the distance of the particle from the j-th axis, i.e. the length of the perpendicular dropped from the point to that axis. The axes of such special coordinate system are called the *principal axes*, while the diagonal elements I_j given by Eq. (24), the *principal moments of inertia* of the body. In such a special reference frame, Eqs. (15) and (22) are reduced to very simple forms:

Rotational energy and angular momentum in principal axes

$$T_{\text{rot}} = \sum_{j=1}^{3} \frac{I_j}{2}\omega_j^2 , \tag{6.25}$$

$$L_j = I_j \omega_j . \tag{6.26}$$

Both these results remind the corresponding relations for the translational motion, $T_{\text{tran}} = MV^2/2$ and $\mathbf{P} = M\mathbf{V}$, with the angular velocity **ω** replacing the "linear" velocity **V**, and the tensor of inertia playing the role of scalar mass M. However, let me emphasize that even in the specially selected coordinate system, with axes pointing in principal directions, the analogy is incomplete, and rotation is generally more complex than translation, because the measures of inertia, I_j, are generally different for each principal axis.

Let me illustrate this fact on a simple but instructive system of three similar massive particles fixed in the vertices of an equilateral triangle (Fig. 3).

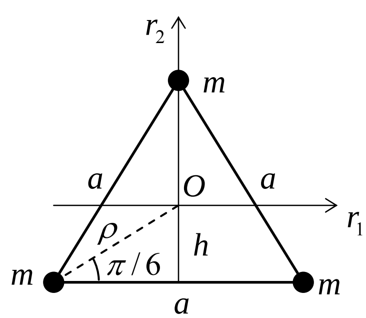

Fig. 6.3. Principal moments of inertia: a simple case study.

Due to symmetry of the configuration, one of the principal axes has to pass through the center of mass O, perpendicular to the plane of the triangle. For the corresponding principal moment of inertia, Eq. (24) readily yields $I_3 = 3m\rho^2$. If we want to express the result in terms of the triangle side a, we may notice that due to system's symmetry, the angle marked in Fig. 3 equals $\pi/6$, and from the corresponding right triangle, $a/2 = \rho\cos(\pi/6) \equiv \rho\sqrt{3}/2$, giving $\rho = a/\sqrt{3}$, so that, finally, $I_3 = ma^2$.

Another way to get the same result is to use the following general *axis shift theorem*, which may be rather useful - especially for more complex cases. Let us relate the inertia tensor components $I_{jj'}$ and $I'_{jj'}$, calculated in two reference frames - one in the center of mass O, and another one displaced by a certain vector $\mathbf{d}$ (Fig. 4a), so that for an arbitrary point, $\mathbf{r}' = \mathbf{r} + \mathbf{d}$. Plugging this relation into Eq. (16), we get

$$\begin{aligned} I'_{jj'} &= \sum m\left[(\mathbf{r}+\mathbf{d})^2\delta_{jj'} - (r_j + d_j)(r_{j'} + d_{j'})\right] \\ &= \sum m\left[\left(r^2 + 2\mathbf{r}\cdot\mathbf{d} + d^2\right)\delta_{jj'} - \left(r_j r_{j'} + r_j d_{j'} + r_{j'} d_j + d_j d_{j'}\right)\right]. \end{aligned} \tag{6.27}$$

Since in the center-of-mass frame, all sums $\sum m r_j$ equal zero, we may use Eq. (16) to finally obtain

$$I'_{jj'} = I_{jj'} + M(\delta_{jj'}d^2 - d_j d_{j'}). \tag{6.28}$$

Rotation axis shift

In particular, this equation shows that if the shift vector $\mathbf{d}$ is perpendicular to one (say, j-th) of the principal axes (Fig. 4b), i.e. $d_j = 0$, then Eq. (28) is reduced to a very simple formula:

$$I'_j = I_j + Md^2. \tag{6.29}$$

Principal axis shift

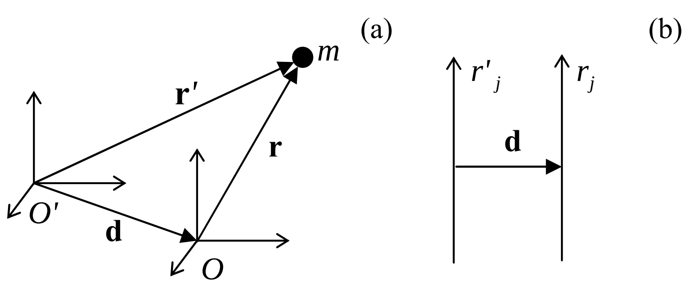

Fig. 6.4. (a) General reference frame shift from the center of mass, and (b) a shift perpendicular to one of the principal axes.

Returning to the system shown in Fig. 3, let us perform such a shift so that the new ("primed") axis passes through the location of one of the particles, still perpendicular to particles' plane. Then the contribution of that particular mass to the primed moment of inertia vanishes, and $I'_3 = 2ma^2$. Now, returning to the center of mass and applying Eq. (29), we get $I_3 = I'_3 - M\rho^2 = 2ma^2 - (3m)(a/\sqrt{3})^2 = ma^2$, i.e. the same result as above.

The symmetry situation inside the triangle plane is somewhat less evident, so let us start with calculating the moments of inertia for the axes shown vertical and horizontal in Fig. 3. From Eq. (24) we readily get:

$$I_1 = 2mh^2 + m\rho^2 = m\left[2\left(\frac{a}{2\sqrt{3}}\right)^2 + \left(\frac{a}{\sqrt{3}}\right)^2\right] = \frac{ma^2}{2}, \qquad I_2 = 2m\left(\frac{a}{2}\right)^2 = \frac{ma^2}{2}, \tag{6.30}$$

where I have taken into account the fact that the distance h from the center of mass and any side of the triangle is $h = \rho\sin(\pi/6) = \rho/2 = a/2\sqrt{3}$. We see that $I_1 = I_2$, and mathematics tells us that in this case *any* in-plane axis (passing through the center of mass O) may be considered as principal, and has the same moment of inertia. A rigid body with this property, $I_1 = I_2 \neq I_3$, is called the *symmetric top*. (The last direction is called the *main principal axis* of the system.)

Symmetric top: definition

Despite the name, the situation may be even more symmetric in the so-called *spherical tops*, i.e. highly symmetric systems whose principal moments of inertia are all equal,

$$I_1 = I_2 = I_3 \equiv I\,, \tag{6.31}$$

Spherical top: definition and description

Mathematics says that in this case the moment of inertia for rotation about *any* axis (but still passing through the center of mass) is equal to the same I. Hence Eqs. (25) and (26) are further simplified for any direction of vector $\boldsymbol{\omega}$:

$$T_{\text{rot}} = \frac{I}{2}\omega^2, \quad \mathbf{L} = I\boldsymbol{\omega}\,, \tag{6.32}$$

thus making the analogy of rotation and translation complete. (As will be discussed in the next section, the analogy is also complete if the rotation axis is fixed by external constraints.)

An evident example of a spherical top is a uniform sphere or spherical shell; a less obvious example is a uniform cube - with masses either concentrated in vertices, or uniformly spread over the faces, or uniformly distributed over the volume. Again, in this case *any* axis passing through the center of mass is principal, and has the same principal moment of inertia. For a sphere, this is natural; for a cube, rather surprising – but may be confirmed by a direct calculation.

6.3. Fixed-axis rotation

Now we are well equipped for a discussion of rigid body's rotational dynamics. The general equation of this dynamics is given by Eq. (1.38), which is valid for dynamics of any system of particles – either rigidly connected or not:

$$\dot{\mathbf{L}} = \boldsymbol{\tau}\,, \tag{6.33}$$

where $\boldsymbol{\tau}$ is the net torque of external forces. Let us start exploring this equation from the simplest case when the axis of rotation, i.e. the direction of vector $\boldsymbol{\omega}$, is fixed by some external constraints. Let us direct axis z along this vector; then $\omega_x = \omega_y = 0$. According to Eq. (22), in this case, the z-component of the angular momentum,

$$L_z = I_{zz}\omega_z\,, \tag{6.34}$$

where I_{zz}, though not necessarily one of the principal momenta of inertia, still may be calculated using Eq. (24):

$$I_{zz} = \sum m\rho_z^2 = \sum m\left(x^2 + y^2\right), \tag{6.35}$$

with ρ_z being the distance of each particle from the rotation axis z. According to Eq. (15), the rotational kinetic energy in this case is just

$$T_{\text{rot}} = \frac{I_{zz}}{2}\omega_z^2 . \quad (6.36)$$

Moreover, it is straightforward to use Eqs. (12), (17), and (28) to show that if the rotation axis is fixed, Eqs. (34)-(36) are valid even if the axis does not pass through the center of mass – if only distances ρ_z are now measured from that axis. (The proof is left for reader's exercise.)

As a result, we may not care about other components of vector **L**,[4] and use just one component of Eq. (33),

$$\dot{L}_z = \tau_z , \quad (6.37)$$

because it, when combined with Eq. (34), completely determines the dynamics of rotation:

$$I_{zz}\dot{\omega}_z = \tau_z , \quad \text{i.e. } I_{zz}\ddot{\theta}_z = \tau_z , \quad (6.38)$$

where θ_z is the angle of rotation about the axis, so that $\omega_z = \dot{\theta}$. Scalar relations (34), (36) and (38), describing rotation about a fixed axis, are completely similar to the corresponding formulas of 1D motion of a single particle, with ω_z corresponding to the usual ("linear") velocity, the angular momentum component L_z - to the linear momentum, and I_z - to particle's mass.

The resulting motion about the axis is also frequently similar to that of a single particle. As a simple example, let us consider what is called the *physical pendulum* (Fig. 5) - a rigid body free to rotate about a fixed horizontal axis *A* that does not pass through the center of mass *O*, in the uniform gravity field **g**.

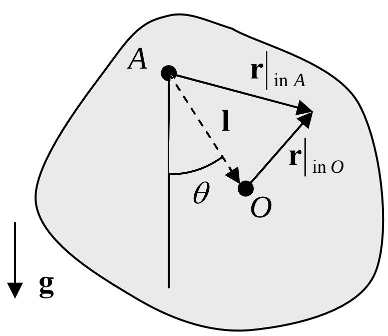

Fig. 6.5. Physical pendulum. The fixed (horizontal) rotation axis *A* is perpendicular to the plane of drawing.

Let us drop a perpendicular from point *O* to the rotation axis, and call the corresponding vector **l** (Fig. 5). Then the torque (relative to axis *A*) of the forces exerted by the axis constraint is zero, and the only contribution to the net torque is due to gravity alone:

$$\boldsymbol{\tau}\big|_{\text{in }A} \equiv \sum \mathbf{r}\big|_{\text{in }A} \times \mathbf{F} = \sum \left(\mathbf{l} + \mathbf{r}\big|_{\text{in }O}\right)\times m\mathbf{g} = \sum m\left(\mathbf{l}\times\mathbf{g}\right) + \sum m\mathbf{r}\big|_{\text{in }O}\times\mathbf{g} = M\mathbf{l}\times\mathbf{g} . \quad (6.39)$$

(For the last transition, I have used the facts that point *O* is the center of mass, and that vectors **l** and **g** are the same for all particles of the body.) This result shows that the torque is directed along the rotation

[4] Note that according to Eq. (22), other Cartesian components of the angular momentum, $L_x = I_{xz}\omega_z$ and $L_y = I_{yz}\omega_z$ may be different from zero, and even evolve in time. (Indeed, if axes *x* and *y* are fixed in lab frame, I_{xz} and I_{yz} may change due to body's rotation.) The corresponding torques $\tau_x^{(\text{ext})}$ and $\tau_y^{(\text{ext})}$, which obey Eq. (33), are automatically provided by external forces which keep the rotation axis fixed.

axis, and its (only) component τ_z is equal to $-Mgl\sin\theta$, where θ is the angle between vectors **l** and **g**, i.e. the angular deviation of the pendulum from the position of equilibrium. As a result, Eq. (38) takes the form,

$$I_A \ddot{\theta} = -Mgl \sin\theta, \tag{6.40}$$

where, I_A is the moment of inertia for rotation about axis A rather about the center of mass. This equation is identical to that of the point-mass (sometimes called "mathematical") pendulum, with the small-oscillation frequency

Physical pendulum's frequency

$$\Omega = \left(\frac{Mgl}{I_A}\right)^{1/2}. \tag{6.41}$$

As a sanity check, in the simplest case when the linear size of the body is much smaller than the suspension length l, Eq. (35) yields $I_A = Ml^2$, and Eq. (41) reduces to the well-familiar formula $\Omega = (g/l)^{1/2}$ for the mathematical pendulum.

Now let us discuss the situations when a body not only rotates, but also moves as the whole. As we already know from our introductory chapter, the total momentum of the body,

$$\mathbf{P} \equiv \sum m\mathbf{v} = \sum m\dot{\mathbf{r}} = \frac{d}{dt}\sum m\mathbf{r}, \tag{6.42}$$

satisfies the 2nd Newton law in the form (1.30). Using the definition (13) of the center of mass, the momentum may be presented as

$$\mathbf{P} = M\dot{\mathbf{R}} = M\mathbf{V}, \tag{6.43}$$

so Eq. (1.30) may be rewritten as

C.o.m.'s law of motion

$$M\dot{\mathbf{V}} = \mathbf{F}, \tag{6.44}$$

where **F** is the vector sum of all external forces. This equation shows that the center of mass of the body moves exactly as a point particle of mass M, under the effect of the net force **F**. In many cases this fact makes the translational dynamics of a rigid body absolutely similar to that of a point particle.

The situation becomes more complex if some of the forces contributing to the vector sum **F** depend on rotation of the same body, i.e. if its rotational and translational motions are coupled. Analysis of such coupled motion is rather straightforward if the *direction* of the rotation axis does not change in time, and hence Eqs. (35)-(36) are still valid. Possibly the simplest example is a round cylinder (say, a wheel) rolling on a surface without slippage (Fig. 6).

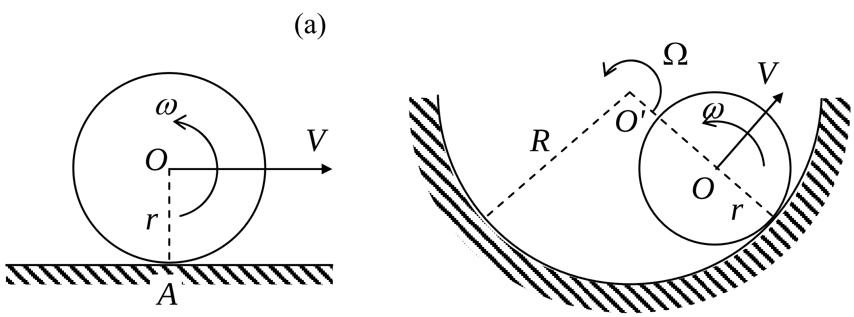

Fig. 6.6. Round cylinder rolling over (a) plane surface and (b) concave surface.

The no-slippage condition may be presented as the requirement of zero net velocity of the particular wheel point A that touches the surface – in the reference frame connected to the surface. For the simplest case of plane surface (Fig. 6a), the application of Eq. (10) shows that this requirement gives the following relation between the angular velocity ω of the wheel and the linear velocity V of its center:

$$V + r\omega = 0. \qquad (6.45)$$

Such kinematic relations are essentially holonomic constraints, which reduce the number of degrees of freedom of the system. For example, without condition (45) the wheel on a plane surface has to be considered as a system with two degrees of freedom, so that its total kinetic energy (14) is a function of two independent generalized velocities, say V and ω:

$$T = T_{\text{tran}} + T_{\text{rot}} = \frac{M}{2}V^2 + \frac{I}{2}\omega^2. \qquad (6.46)$$

Using Eq. (45) we may eliminate, for example, the linear velocity and reduce Eq. (46) to

$$T = \frac{M}{2}(\omega r)^2 + \frac{I}{2}\omega^2 = \frac{I_{\text{ef}}}{2}\omega^2, \quad \text{where} \quad I_{\text{ef}} \equiv I + Mr^2. \qquad (6.47)$$

This result may be interpreted as the kinetic energy of pure rotation of the wheel about the instantaneous axis A, with I_{ef} being the moment of inertia about that axis, satisfying Eq. (29).

Kinematic relations are not always as simple as Eq. (45). For example, if the wheel is rolling on a concave surface (Fig. 6b), we need relate the angular velocities of the wheel rotation about its axis O (denoted ω) and that of its axis' rotation about the center O' of curvature of the surface (Ω). A popular error here is to write $\Omega = -(r/R)\omega$ [WRONG!]. A prudent way to get the correct relation is to note that Eq. (45) holds for this situation as well, and on the other hand the same linear velocity of wheel's center may be expressed as $V = (R - r)\Omega$. Combining these equations, we get a (not quite evident) relation

$$\Omega = -\frac{r}{R - r}\omega. \qquad (6.48)$$

Another famous example of the relation between the translational and rotational motion is given by the "sliding ladder" problem (Fig. 7). Let us analyze it for the simplest case of negligible friction, and ladder's thickness small in comparison with its length l.

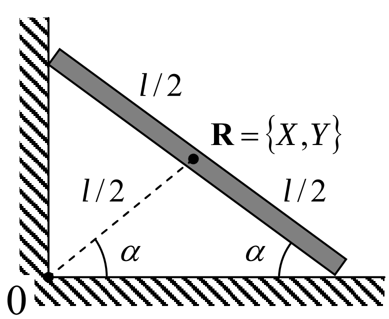

Fig. 6.7. Sliding ladder problem.

In order to use the Lagrangian formalism, we may write the kinetic energy of the ladder as the sum (14) of the translational and rotational parts:

$$T = \frac{M}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{I}{2}\dot{\alpha}^2, \tag{6.49}$$

where X and Y are the Cartesian coordinates of its center of mass in an inertial reference frame, and I is the moment of inertia for rotation about the z-axis passing through the center of mass. (For the uniformly-distributed mass, an elementary integration of Eq. (35) yields $I = Ml^2/12$). In the reference frame with the center in the corner O, both X and Y may be simply expressed via angle α :

$$X = \frac{l}{2}\cos\alpha, \quad Y = \frac{l}{2}\sin\alpha. \tag{6.50}$$

(The easiest way to obtain these relations is to notice that the dashed line in Fig. 7 has slope α and length $l/2$.) Plugging these expressions into Eq. (49), we get

$$T = \frac{I_{\text{ef}}}{2}\dot{\alpha}^2, \quad I_{\text{ef}} \equiv I + M\left(\frac{l}{2}\right)^2 = \frac{1}{3}Ml^2. \tag{6.51}$$

Since the potential energy of the ladder in the gravity field may be also expressed via the same angle,

$$U = MgY = Mg\frac{l}{2}\sin\alpha, \tag{6.52}$$

α may be conveniently used as the (only) generalized coordinate of the system. Even without writing the Lagrangian equation of motion for that coordinate explicitly, we may notice that since the Lagrangian function ($T - U$) does not depend on time explicitly, and the kinetic energy (51) is a quadratic-homogeneous function of the generalized velocity $\dot{\alpha}$, the full mechanical energy,

$$E \equiv T + U = \frac{I_{\text{ef}}}{2}\dot{\alpha}^2 + Mg\frac{l}{2}\sin\alpha = \frac{Mgl}{2}\left(\frac{l\dot{\alpha}^2}{3g} + \sin\alpha\right), \tag{6.53}$$

is conserved and gives us the first integral of motion. Moreover, Eq. (53) shows that the system's energy (and hence dynamics) is identical to that of a physical pendulum with an unstable fixed point $\alpha_1 = \pi/2$, stable fixed point at $\alpha_2 = -\pi/2$, and frequency

$$\Omega = \left(\frac{3g}{2l}\right)^{1/2} \tag{6.54}$$

of small oscillations near the latter point. (Of course, that fixed point cannot be reached in the simple geometry shown in Fig. 7, where ladder's hitting the floor would change its equations of motion).

6.4. Free rotation

Now let us proceed to more complex case when the rotation axis is *not* fixed. A good illustration of the complexity arising is this case comes from the simplest case of a rigid body left alone, i.e. not subjected to external forces and hence its potential energy U is constant. Since in this case, according to Eq. (44), the center of mass moves (as measured from any inertial reference frame) with a constant velocity, we can always use an convenient inertial reference frame with the center at that point. From the point of view of such frame, the body's motion is a pure rotation, and $T_{\text{tran}} = 0$. Hence, the system's Lagrangian equals just the rotational energy (15), which is, first, a quadratic-homogeneous function of components ω_j (that may be taken for generalized velocities), and, second, does not depend on time explicitly. As we know from Chapter 2, in this case the energy is conserved. For the components of vector $\boldsymbol{\omega}$ in the principal axes, this means

$$T_{\text{rot}} = \sum_{j=1}^{3} \frac{I_j}{2} \omega_j^2 = \text{const}\,. \tag{6.55}$$

Rotational energy's conservation

Next, as Eq. (33) shows, in the absence of external forces the angular momentum $\mathbf{L}$ of the body is conserved as well. However, though we can certainly use Eq. (26) to present this fact as

$$\mathbf{L} = \sum_{j=1}^{3} I_j \omega_j \mathbf{n}_j = \text{const}\,, \tag{6.56}$$

Angular momentum's conservation

where $\mathbf{n}_j$ are the principal axes of inertia, this does not mean that components ω_j of the angular velocity vector $\boldsymbol{\omega}$ are constant, because the principal axes are fixed relative to the rigid body, and hence may rotate with it.

Before going after these complications, let us briefly mention two conceptually trivial, but practically very important, particular cases. The first is a spherical top ($I_1 = I_2 = I_3 = I$). In this case Eqs. (55) and (56) imply that all components of vector $\boldsymbol{\omega} = \mathbf{L}/I$, i.e. both the magnitude and the direction of the angular velocity are conserved, for any initial spin. In other words, the body conserves its rotation speed and axis direction, as measured in an inertial frame.

The most obvious example is a spherical planet. For example, our Mother Earth, rotating about its axis with angular velocity $\omega = 2\pi/(1 \text{ day}) \approx 7.3\times10^{-5}\ \text{s}^{-1}$, keeps its axis at a nearly constant angle of 23°27' to the *ecliptic pole*, i.e. the axis normal to the plane of its motion around the Sun. (In Sec. 6 below, we will discuss some very slow motions of this axis, due to gravity effects.)

Spherical tops are also used in the most accurate gyroscopes, usually with gas or magnetic suspension in vacuum. If done carefully, such systems may have spectacular stability. For example, the gyroscope system of the Gravity Probe B satellite experiment, flown in 2004-2005, was based on quartz spheres - round with precision of about 10 nm and covered by superconducting thin films (which have enabled their magnetic suspension and SQUID monitoring). The whole system was stable enough to measure that the so-called *geodetic effect* in general relativity (essentially, the space curving by Earth's mass), resulting in the axis precession by just 6.6 arcseconds per year, i.e. with a precession frequency of just $\sim10^{-11}\text{s}^{-1}$, agrees with theory with a record ~0.3% accuracy.[5]

The second simple case is that of the "symmetric top" ($I_1 = I_2 \neq I_3$), with the initial vector $\mathbf{L}$ aligned with the main principal axis. In this case, $\boldsymbol{\omega} = \mathbf{L}/I_3 = \text{const}$, so that the rotation axis is conserved.[6] Such tops, typically in the shape of a flywheel (rotor) supported by a "gimbal" system (Fig. 8), are broadly used in more common gyroscopes, core parts of automatic guidance systems, for

[5] Such beautiful experimental physics does not come cheap: the total Gravity Probe B project budget was about $750M. Even at this price tag, the declared main goal of the project, an accurate measurement of a more subtle relativistic effect, the so-called *frame-dragging drift* (or "the Schiff precession"), predicted to be about 0.04 arc seconds per year, has not been achieved.

[6] This is also true for an asymmetric top, i.e. an arbitrary body (with, say, $I_1 < I_2 < I_3$), but in this case the alignment of vector $\mathbf{L}$ with axis $\mathbf{n}_2$, corresponding to the intermediate moment of inertia, is unstable.

example, in ships, airplanes, missiles, etc. Even if the ship's hull wobbles, the suspended gyroscope sustains its direction relative to Earth (which is sufficiently inertial for these applications).[7]

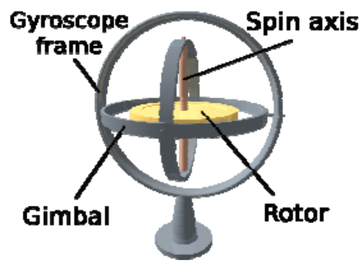

Fig. 6.8. Typical gyroscope. (Adapted from http://en.wikipedia.org/wiki/Gyroscope.)

However, in the general case with no such special initial alignment, the dynamics of symmetric tops is more complex. In this case, vector **L** is still conserved, including its direction, but vector **ω** is not. Indeed, let us direct axis $\mathbf{n}_2$ perpendicular to the common plane of vectors **L** and the instantaneous direction $\mathbf{n}_3$ of the main principal axis (in Fig. 9, the plane of drawing); then, in that particular instant, $L_2 = 0$. Now let us recall that in a symmetric top, axis $\mathbf{n}_2$ is a principal one. According to Eq. (26) with $j = 2$, the corresponding component ω_2 has to be equal to L_2/I_2, so it vanishes. This means that vector **ω** lies in this plane (the common plane of vectors **L** and $\mathbf{n}_3$) as well – see Fig. 9a.

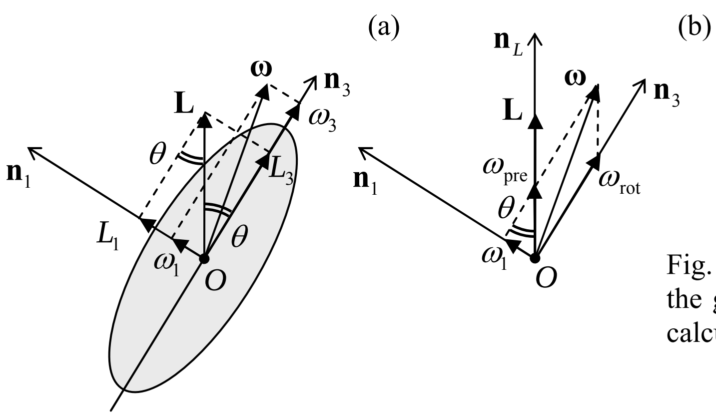

Fig. 6.9. Free rotation of a symmetric top: (a) the general configuration of vectors, and (b) calculating the free precession frequency.

Now consider any point of the body, located on axis $\mathbf{n}_3$, and hence within plane $[\mathbf{n}_3, \mathbf{L}]$. Since **ω** is the instantaneous axis of rotation, according to Eq. (9), the point has instantaneous velocity $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ directed normally to that plane. Since this is true for each point of the main axis (besides only one, with $\mathbf{r} = 0$, i.e. the center of mass, which does not move), this axis as a whole has to move perpendicular to the common plane of vectors **L**, **ω**, and $\mathbf{n}_3$. Since such conclusion is valid for any moment of time, it means that vectors **ω** and $\mathbf{n}_3$ rotate about the space-fixed vector **L** together, with some angular velocity ω_{pre}, at each moment staying in one plane. This effect is usually called the *free precession* (or "torque-

[7] Much more compact (and much less accurate) gyroscopes used, e.g., in smartphones and tablet computers, are based on the effect of rotation on oscillator frequency, and implemented as micro-electromechanical systems (MEMS) on silicon chip surface – see, e.g., Chapter 22 in V. Kaajakari, *Practical MEMS*, Small Gear Publishing, 2009.

free", or "regular") *precession*, and has to be clearly distinguished it from the completely different effect of the *torque-induced precession* which will be discussed in the next section.

In order to calculate ω_{pre}, let us present the instant vector $\boldsymbol{\omega}$ as a sum of not its Cartesian coordinates (as in Fig. 9a), but rather of two non-orthogonal vectors directed along $\mathbf{n}_3$ and $\mathbf{L}$ (Fig. 9b):

$$\boldsymbol{\omega} = \omega_{rot}\mathbf{n}_3 + \omega_{pre}\mathbf{n}_L, \quad \mathbf{n}_L \equiv \frac{\mathbf{L}}{L}. \tag{6.57}$$

It is clear from Fig. 9b that ω_{rot} has the meaning of the angular velocity of body rotation of the body about its main principal axis, while ω_{pre} is the angular velocity of rotation of that axis about the constant direction of vector **L,** i.e. the frequency of precession. Now the latter frequency may be readily calculated from the comparison of two panels of Fig. 9, by noticing that the same angle θ between vectors **L** and $\mathbf{n}_3$ participates in two relations:

$$\sin\theta = \frac{L_1}{L} = \frac{\omega_1}{\omega_{pre}}. \tag{6.58}$$

Since axis $\mathbf{n}_1$ is principal, we may use Eq. (26) for $j = 1$, i.e. $L_1 = I_1\omega_1$, to eliminate ω_1 from Eq. (58), and get a very simple formula

$$\omega_{pre} = \frac{L}{I_1}. \tag{6.59}$$

Free precession frequency

This result shows that the precession *frequency* is constant and independent of the alignment of vector **L** with the main principal axis $\mathbf{n}_3$, while the *amplitude* of this motion (characterized by angle θ) does depend on the alignment, and vanishes if **L** is parallel to $\mathbf{n}_3$.[8] Note also that if all principal moments of inertia are of the same order, ω_{pre} is of the same order as the total angular velocity $\omega = |\boldsymbol{\omega}|$ of rotation.

Now, let us briefly discuss the free precession in the general case of an "asymmetric top", i.e. a body with $I_1 \neq I_2 \neq I_3$. In this case the effect is more complex because here not only the *direction* but also the *magnitude* of the instantaneous angular velocity $\boldsymbol{\omega}$ may evolve in time. If we are only interested in the relation between the instantaneous values of ω_j and L_j, i.e. the "trajectories" of vectors $\boldsymbol{\omega}$ and **L** as observed from the reference frame $\{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\}$ of the principal axes of the body (rather than an explicit law of their time evolution), they may be found directly from the conservation laws. (Let me emphasize again that vector **L**, being constant in an inertial frame, generally evolves in the frame rotating with the body.) Indeed, Eq. (55) may be understood as the equation of an ellipsoid in Cartesian coordinates $\{\omega_1, \omega_2, \omega_3\}$, so that for free body, vector $\boldsymbol{\omega}$ has to stay on the surface of that ellipsoid.[9] On the other hand, since the reference frame rotation preserves the length of any vector, the *magnitude* (but not direction!) of vector **L** is also an integral of motion in the moving frame, and we can write

$$L^2 \equiv \sum_{j=1}^{3} L_j^2 = \sum_{j=1}^{3} I_j^2\omega_j^2 = \text{const}. \tag{6.60}$$

[8] For Earth, the free precession amplitude it so small (below 10 m of linear displacement on the Earth surface) that this effect is of the same order as other, irregular motions of the rotation axis, resulting from the turbulent fluid flow effects in planet's interior and its atmosphere.

[9] It is frequently called the *Poinsot ellipsoid*, after L. Poinsot (1777-1859) who has made several pioneering contributions to the rigid body mechanics.

Hence the trajectory of vector **ω** follows the closed curve formed by the intersection of two ellipsoids, (55) and (60). It is evident that this trajectory is generally "taco-edge-shaped", i.e. more complex than a plane circle but never very complex either.

The same argument may be repeated for vector **L**, for whom the first form of Eq. (60) descries a sphere, and Eq. (55), another ellipsoid:

$$T_{\text{rot}} = \sum_{j=1}^{3} \frac{1}{2I_j} L_j^2 = \text{const}. \tag{6.61}$$

On the other hand, if we are interested in the trajectory of vector **ω** in an inertial frame (in which vector **L** stays still), we may note that the general relation (15) for the same rotational energy T_{rot} may also be rewritten as

$$T_{\text{rot}} = \frac{1}{2}\sum_{j=1}^{3}\omega_j \sum_{j'=1}^{3} I_{jj'}\omega_{j'}. \tag{6.62}$$

But according to the Eq. (22), the second sum in the right-hand part is nothing more than L_j, so that

$$T_{\text{rot}} = \frac{1}{2}\sum_{j=1}^{3}\omega_j L_j = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}. \tag{6.63}$$

This equation shows that for a free body (T_{rot} = const, **L** = const), even is vector **ω** changes in time, its end point should stay within a plane perpendicular to angular momentum **L**. (Earlier, we have seen that for the particular case of the symmetric top – see Fig. 9b, but for an asymmetric top, the trajectory of the end point may not be circular.)

If we are interested not only in the trajectory of vector **ω**, but also its explicit evolution in time, it may be calculated using the general Eq. (33) presented in principal components ω_j. For that, we have to recall that Eq. (33) is only valid in an inertial reference frame, while the frame {$\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$} may rotate with the body and hence is generally not inertial. We may handle this problem by applying to vector **L** the general relation (8):

$$\left.\frac{d\mathbf{L}}{dt}\right|_{\text{in lab}} = \left.\frac{d\mathbf{L}}{dt}\right|_{\text{in mov}} + \boldsymbol{\omega}\times\mathbf{L}. \tag{6.64}$$

Combining it with Eq. (33), in the moving frame we get

$$\frac{d\mathbf{L}}{dt} + \boldsymbol{\omega}\times\mathbf{L} = \boldsymbol{\tau}, \tag{6.65}$$

where **τ** is the external torque. In particular, for the principal-axis components L_j, related to components ω_j by Eq. (26), Eq. (65) is reduced to a set of three scalar *Euler equations*

Euler equations

$$I_j\dot{\omega}_j + (I_{j''} - I_{j'})\omega_{j'}\omega_{j''} = \tau_j, \tag{6.66}$$

where the set of indices { *j*, *j'* , *j"* } has to follow the usual "right" order - e.g., {1, 2, 3}, etc.[10]

[10] These equations are of course valid in the simplest case of the fixed rotation axis as well. For example, if $\boldsymbol{\omega} = \mathbf{n}_z\omega$, i.e. $\omega_x = \omega_y = 0$, Eq. (66) is reduced to Eq. (38).

In order to get a feeling how do the Euler equations work, let us return to the case of a free symmetric top ($\tau_1 = \tau_2 = \tau_3 = 0$, $I_1 = I_2 \neq I_3$). In this case, $I_1 - I_2 = 0$, so that Eq. (66) with $j = 3$ yields ω_3 = const, while the equations for $j = 1$ and $j = 2$ take the simple form

$$\dot{\omega}_1 = -\Omega_{\text{pre}}\omega_2, \quad \dot{\omega}_2 = \Omega_{\text{pre}}\omega_1, \tag{6.67}$$

where Ω_{pre} is a constant determined by the system parameters and initial conditions:

$$\Omega_{\text{pre}} \equiv \omega_3 \frac{I_3 - I_1}{I_1}. \tag{6.68}$$

Obviously, Eqs. (67) have a sinusoidal solution with frequency Ω_{pre}, and describe uniform rotation of vector $\boldsymbol{\omega}$, with that frequency, about the main axis $\mathbf{n}_3$. This is just another presentation of the torque-free precession analyzed above, this time as observed from the rotating body. Evidently, Ω_{pre} is substantially different from the frequency ω_{pre} (59) of the precession as observed from the lab frame; for example, the former frequency vanishes for the spherical top (with $I_1 = I_2 = I_3$), while the latter frequency tends to the rotation frequency.

Unfortunately, for the rotation of an asymmetric top (i.e., an arbitrary rigid body), when no component ω_j is conserved, the Euler equations (66) are strongly nonlinear even in the absence of the external torque, and a discussion of their solutions would take more time than I can afford.[11]

6.5. Torque-induced precession

The dynamics of rotation becomes even more complex in the presence of external forces. Let us consider the most important and counter-intuitive effect of *torque-induced precession*, for the simplest case of an axially-symmetric body (which is a particular case of the symmetric top, $I_1 = I_2 \neq I_3$) rapidly spinning about his symmetry axis, and supported at some point A of that axis, that does not coincide with the center of mass O – see Fig. 10. Without external forces, such top would retain the direction of its rotation axis that would always coincide with the direction of the angular momentum:

$$\mathbf{L} = I_3\boldsymbol{\omega} = I_3\omega_{\text{rot}}\mathbf{n}_3. \tag{6.69}$$

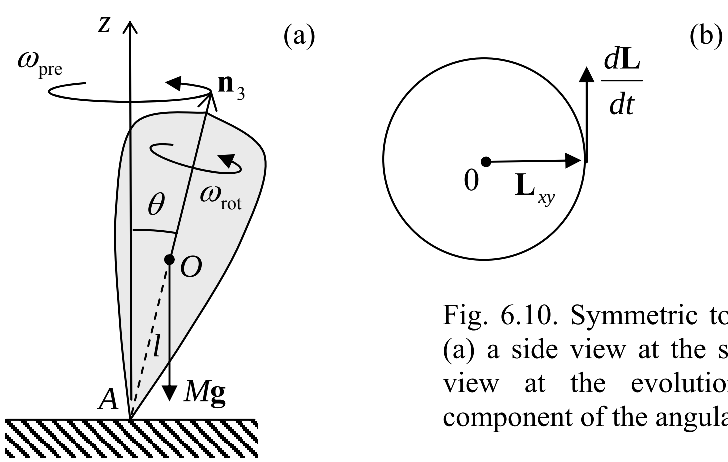

Fig. 6.10. Symmetric top in the gravity field: (a) a side view at the system and (b) the top view at the evolution of the horizontal component of the angular momentum vector.

[11] Such discussion may be found, e.g. in Sec. 37 of L. Landau and E. Lifshitz, *Mechanics*, 3rd ed., Butterworth-Heinemann, 1976.

The uniform gravity field creates bulk-distributed forces that, as we know from the analysis of the physical pendulum in Sec. 3, are equivalent to a single force $M\mathbf{g}$ applied in the center of mass – in Fig. 10, point O. The torque of the force relative to the support point A is

$$\boldsymbol{\tau} = \mathbf{r}_O\big|_{\text{in } A} \times M\mathbf{g} = Ml\mathbf{n}_3 \times \mathbf{g}. \tag{6.70}$$

Hence the general equation (33) of the angular momentum (valid in the inertial "lab" frame, in which point A rests) becomes

Precession: equation

$$\dot{\mathbf{L}} = Ml\mathbf{n}_3 \times \mathbf{g}. \tag{6.71}$$

Despite the apparent simplicity of this (exact!) equation, its analysis is straightforward only in the limit of relatively high rotation velocity ω_{rot} or, alternatively, very small torque. In this limit, we may, in the 0th approximation, still use Eq. (69) for **L.** Then Eq. (71) shows that vector $\dot{\mathbf{L}}$ is perpendicular to both $\mathbf{n}_3$ (and hence **L**) and **g**, i.e. lies within the horizontal plane, and is perpendicular to the horizontal component $\mathbf{L}_{xy}$ of vector **L** – see Fig. 10b. Since the magnitude of this vector is constant, $|\dot{\mathbf{L}}| = mgl\sin\theta$, vector **L** (and hence the body's main axis) rotates about the vertical axis with angular velocity

Precession: frequency

$$\omega_{\text{pre}} = \frac{|\dot{\mathbf{L}}|}{L_{xy}} = \frac{Mgl\sin\theta}{L\sin\theta} = \frac{Mgl}{L} = \frac{Mgl}{I_3\omega_{\text{rot}}}. \tag{6.72}$$

Thus, very counter-intuitively, the fast-rotating top "does not want to" follow the external, vertical force and, in addition to fast spinning about the symmetry axis $\mathbf{n}_3$, also performs a revolution, called the *torque-induced precession*, about the vertical axis. Note that, similarly to the free-precession frequency (59), the torque-induced precession frequency (72) does not depend on the initial (and sustained) angle θ. However, the torque-induced precession frequency is inversely (rather than directly) proportional to ω, and is typically much lower. This relative slowness is also required for the validity of our simple theory of this effect. Indeed, in our approximate treatment we have used Eq. (69), i.e. neglected precession's contribution to the angular momentum vector **L**. This is only possible if the contribution is relatively small, $I\omega_{\text{pre}} << I_3\omega_{\text{rot}}$, where I is a certain effective moment of inertia for the precession (to be worked out later). Using our result (72), this condition may be rewritten as

$$\omega_{\text{rot}} >> \left(\frac{MglI}{I_3^2}\right)^{1/2}. \tag{6.73}$$

For a body of not too extreme proportions, i.e. with all linear dimensions of the order of certain length l, all inertia moments are of the order of Ml^2, so that the right-hand part of Eq. (73) is of the order of $(g/l)^{1/2}$, i.e. comparable with the eigenfrequency of the same body as the physical pendulum, i.e. at the absence of fast rotation.

In order to develop a qualitative theory that could take us beyond such approximate treatment, the Euler equations (66) may be used, but are not very convenient. A better approach, suggested by the same L. Euler, is to introduce a set of three independent angles between the principal axes $\{\mathbf{n}_1,\mathbf{n}_2,\mathbf{n}_3\}$ bound to the rigid body, and axes $\{\mathbf{n}_x,\mathbf{n}_y,\mathbf{n}_z\}$ of an inertial reference frame (Fig. 11), and then express the basic equation (33) of rotation, via these angles. There are several possible options for the definition of such angles;[12] Fig. 11 shows the set of *Euler angels*, most convenient for discussion of fast rotation. As one can see at the figure, the first Euler angle, θ, is the usual polar angle measured from axis $\mathbf{n}_z$ to axis $\mathbf{n}_3$. The second one is the azimuthal angle φ, measured from axis $\mathbf{n}_x$ to the so-called *line of nodes* formed by the intersection of planes $[\mathbf{n}_x,\mathbf{n}_y]$ and $[\mathbf{n}_1,\mathbf{n}_2]$. The last Euler angle, ψ, is measured within plane $[\mathbf{n}_1,\mathbf{n}_2]$, from the line of nodes to axis $\mathbf{n}_1$. In the simple picture of the force-induced precession of a symmetric top, which was derived above, angle θ is constant, angle ψ changes very rapidly, with the rotation velocity ω_{rot}, while angle φ grows with the precession frequency ω_{pre} (72).

Euler angles

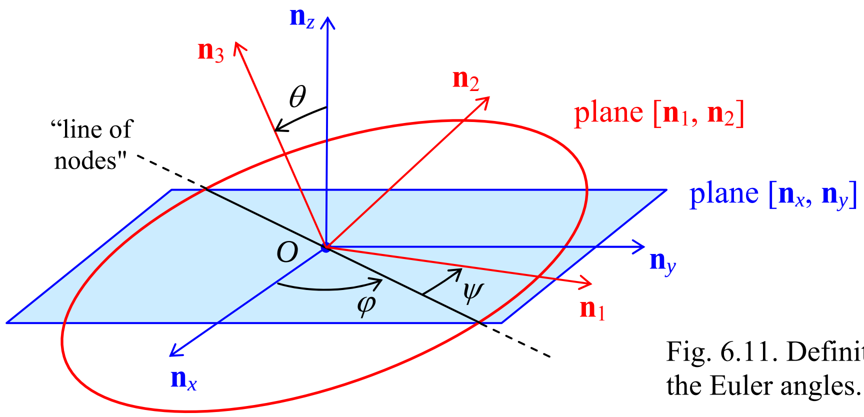

Fig. 6.11. Definition of the Euler angles.

Now we can express the principal-axes components of the instantaneous angular velocity vector, ω_1, ω_2, and ω_3, as measured in the lab reference frame, in terms of the Euler angles. It may be easily done calculating, from Fig. 11, the contributions to the change of Euler angles to each principal axis, and then adding them up. The result is

$$\begin{aligned} \omega_1 &= \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi, \\ \omega_2 &= \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi, \\ \omega_3 &= \dot{\varphi}\cos\theta + \dot{\psi}. \end{aligned} \tag{6.74}$$

Components of ω via Euler angles

These formulas allow the expression of the kinetic energy of rotation (25) and the angular momentum components (26) in terms of the generalized coordinates θ, φ, and ψ, and use then powerful Lagrangian formalism to derive their equations of motion. This is especially simple to do in the case of symmetric tops (with $I_1 = I_2$), because plugging Eqs. (74) into Eq. (25) we get an expression,

$$T_{\text{rot}} = \frac{I_1}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)^2, \tag{6.75}$$

which does not include explicitly either φ or ψ. (This reflects the fact that for a symmetric top we can always select axis $\mathbf{n}_1$ to coincide with the line of nodes, and hence take $\psi = 0$ at the considered moment of time. Note that this trick does *not* mean we can take $\dot{\psi} = 0$, because axis $\mathbf{n}_1$, as observed from the inertial reference frame, moves!) Now we should not forget that at the torque-induced precession, the center of mass moves as well (see Fig. 10), so that according to Eq. (14), the total kinetic energy of the body is the sum of two terms,

[12] Of the several choices more convenient in the absence of fast rotation, the most common is the set of so-called *Tait-Brian angles* (called the *yaw*, *pitch*, and *roll*) that are broadly used in airplane and maritime navigation.

$$T = T_{\text{rot}} + T_{\text{tran}}, \quad T_{\text{tran}} = \frac{M}{2}V^2 = \frac{M}{2}l^2\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta\right), \tag{6.76}$$

while the potential energy is just

$$U = Mgl\cos\theta + \text{const}. \tag{6.77}$$

Now we could readily write the Lagrangian equations of motion for the Euler angles, but it is better to immediately notice that according to Eqs. (75)-(77), the Lagrangian function, $T - U$, does not depend explicitly on "cyclic" coordinates φ and ψ, so that the corresponding generalized momenta are conserved:

$$p_\varphi \equiv \frac{\partial T}{\partial \dot{\varphi}} = I_A \dot{\varphi}\sin^2\theta + I_3(\dot{\varphi}\cos\theta + \dot{\psi})\cos\theta = \text{const}, \tag{6.78}$$

$$p_\psi \equiv \frac{\partial T}{\partial \dot{\psi}} = I_3(\dot{\varphi}\cos\theta + \dot{\psi}) = \text{const}, \tag{6.79}$$

where, according to Eq. (29), $I_A \equiv I_1 + Ml^2$ is just the body's moment of inertia for rotation about a horizontal axis passing through the support point A. According to the last of Eqs. (74), p_ψ is just L_3, the angular momentum's component along the *rotating* axis $\mathbf{n}_3$. On the other hand, by its definition p_φ is L_z, the same vector $\mathbf{L}$'s component along the *static* axis z. (Actually, we could foresee in advance the conservation of both these components of $\mathbf{L}$, because vector (70) of the external torque is perpendicular to both $\mathbf{n}_3$ and $\mathbf{n}_z$.) Using these notions, and solving the simple system of linear equations (78)-(79) for the angle derivatives, we get

$$\dot{\varphi} = \frac{L_z - L_3\cos\theta}{I_A \sin^2\theta}, \quad \dot{\psi} = \frac{L_3}{I_3} - \frac{L_z - L_3\cos\theta}{I_A\sin^2\theta}\cos\theta. \tag{6.80}$$

One more conserved quantity in this problem is the full mechanical energy[13]

$$E \equiv T + U = \frac{I_A}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\varphi}\cos\theta + \dot{\psi}\right)^2 + Mgl\cos\theta. \tag{6.81}$$

Plugging Eqs. (80) into Eq. (81), we get a first-order differential equation for angle θ, which may be presented in the following physically transparent form:

$$\frac{I_A}{2}\dot{\theta}^2 + U_{\text{ef}}(\theta) = E, \qquad U_{\text{ef}}(\theta) \equiv \frac{(L_z - L_3\cos\theta)^2}{2I_A\sin^2\theta} + \frac{L_3^2}{2I_3} + Mgl\cos\theta + \text{const}. \tag{6.82}$$

Thus, similarly to the planetary problems considered in Sec. 3.5, the symmetric top precession has been reduced (without any approximations!) to a 1D problem of motion of one of its degrees of freedom, the polar angle θ, in an effective potential $U_{\text{ef}}(\theta)$, which is the sum of the real potential energy U (77) and a contribution from the kinetic energy of motion along two other angles. In the absence of rotation about axes $\mathbf{n}_z$ and $\mathbf{n}_3$ (i.e., $L_z = L_3 = 0$), Eq. (82) is reduced to the first integral of the equation (40) of motion of a physical pendulum. If the rotation is present, then (besides the case of special initial

[13] Indeed, since the Lagrangian does not depend on time explicitly, H = const, and since the full kinetic energy T is a quadratic-homogeneous function of the generalized velocities, $E = H$.

conditions when $\theta(0) = 0$ *and* $L_z = L_3$),[14] the first contribution to $U_{ef}(\theta)$ diverges at $\theta \to 0$ and π, so that the effective potential energy has a minimum at some finite value θ_0 of the polar angle θ.

If the initial angle $\theta(0)$ equals this θ_0, i.e. if the initial effective energy is equal to its minimum value $U_{ef}(\theta_0)$, the polar angle remains constant through the motion: $\theta(t) = \theta_0$. This corresponds to the pure torque-induced precession whose angular velocity is given by the first of Eqs. (80):

$$\omega_{pre} \equiv \dot{\varphi} = \frac{L_z - L_3 \cos\theta_0}{I_A \sin^2 \theta_0}. \tag{6.83}$$

The condition for finding θ_0, $dU_{ef}/d\theta = 0$, is a transcendent algebraic equation that cannot be solved analytically for arbitrary parameters. However, in the high spinning speed limit (73), this is possible. Indeed, in this limit the potential energy contribution to U_{ef} is small, and we may analyze its effect by successive approximations. In the 0th approximation, i.e. at $Mgl = 0$, the minimum of U_{ef} is evidently achieved at $\cos\theta_0 = L_z/L_3$, giving zero precession frequency (83). In the next, 1st approximation, we may require that at $\theta = \theta_0$, the derivative of first term in the right-hand part of Eq. (82) for U_{ef} over $\cos\theta$, equal to $-L_z(L_z - L_3\cos\theta)/I_A\sin^2\theta$,[15] is cancelled with that of the gravity-induced term, equal to Mgl. This immediately yields $\omega_{pre} = (L_z - L_3\cos\theta_0)/I_A\sin^2\theta_0 = Mgl/L_3$, so that taking $L_3 = I_3\omega_{rot}$ (as we may in the high spinning speed limit), we recover the simple expression (72).

The second important result that readily follows from Eq. (82) is the exact expression the threshold value of the spinning speed for a vertically rotating top ($\theta = 0$, $L_z = L_3$). Indeed, in the limit $\theta \to 0$ this expression may be readily simplified:

$$U_{ef}(\theta) \approx \text{const} + \left(\frac{L_3^2}{8I_A} - \frac{Mgl}{2}\right)\theta^2. \tag{6.84}$$

This formula shows that if $\omega_3 = L_3/I_3$ (i.e. the angular velocity that was called ω_{rot} in the approximate theory) is higher than the following threshold value,

$$\omega_{th} \equiv 2\left(\frac{MglI_A}{I_3^2}\right)^{1/2}, \tag{6.85}$$

Threshold angular velocity

then the coefficient at θ^2 in Eq. (84) is positive, so that U_{ef} has a stable minimum at $\theta_0 = 0$. On the other hand, if ω_3 is decreased below ω_{th}, the fixed point becomes unstable, so that the top falls down. Note that if we take $I = I_A$ in condition (73) of the approximate treatment, it acquires a very simple sense: $\omega_{rot} >> \omega_{th}$.

Finally, Eqs. (82) give a natural description of one more phenomenon. If the initial energy is larger than $U_{ef}(\theta_0)$, angle θ oscillates between two classical turning points on both sides of the fixed point θ_0. The law and frequency of these oscillations may be found exactly as in Sec. 3.3 – see Eqs. (3.27) and (3.28). At $\omega_3 >> \omega_{th}$, this motion is a fast rotation of the symmetry axis $\mathbf{n}_3$ of the body about its average position performing the slow precession. These oscillations are called *nutations*, but

[14] In that simple case the body continues to rotate about the vertical symmetry axis: $\theta(t) = 0$. Note, however, that such motion is stable only if the spinning speed is sufficiently high – see below.

[15] Indeed, the derivative of the fraction $1/2I_A\sin^2\theta$, taken at the point $\cos\theta = L_z/L_3$, is multiplied by the nominator, $(L_z - L_3\cos\theta)^2$, which at this point vanishes.

physically they are absolutely similar to the free precession that was analyzed in the previous section, and the order of magnitude of their frequency is still given by Eq. (59).

It may be proved that small energy dissipation (not taken into account in our analysis) leads first to a decay of nutations, then to a slower drift of the precession angle θ_0 to zero and, finally, to a gradual decay of the spinning speed ω_3 until it reaches the threshold (85) and the top falls down.

6.6. Non-inertial reference frames

Before moving on to the next chapter, let us use the results of our discussion of rotation kinematics in Sec. 1 to complete the analysis of transfer between two reference frames, started in the introductory Chapter 1 – see Fig. 1.2. Indeed, the differentiation rule described by Eq. (8) and derived for an arbitrary vector **A** enables us to relate not only radius-vectors, but also the velocities and accelerations of a particle as measured in two reference frames: the "lab" frame *O'* (which will be later assumed inertial) and the "moving" (possibly rotating) frame *O* – see Fig. 12.

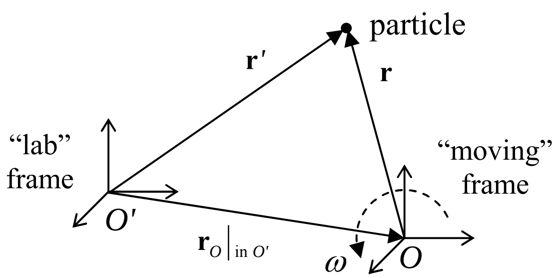

Fig. 6.12. General case of transfer between two reference frames.

As this picture shows, even if frame *O* rotates relative to the lab frame, the radius-vectors are still related, at any moment of time, by the simple Eq. (1.7). In the notation of Fig. 12 it reads

$$\mathbf{r}'\big|_{\text{in lab}} = \mathbf{r}_O\big|_{\text{in lab}} + \mathbf{r}\big|_{\text{in lab}} . \tag{6.86}$$

However, as was discussed in Sec. 1, for velocities the general addition rule is already more complex. In order to find it, let us differentiate Eq. (86) over time:

$$\frac{d}{dt}\mathbf{r}'\Big|_{\text{in lab}} = \frac{d}{dt}\mathbf{r}_O\Big|_{\text{in lab}} + \frac{d}{dt}\mathbf{r}\Big|_{\text{in lab}} . \tag{6.87}$$

The left-hand part of this relation is evidently particle's velocity as measured in the lab frame, and the first term in the right-hand part of Eq. (87) is the velocity of point *O*, as measured in the same frame. The last term is more complex: we need to differentiate vector **r** that connects point *O* with the particle (Fig. 12), considering how its evolution looks from the *lab* frame. Due to the possible mutual rotation of frames *O* and *O'*, that term may not be zero even if the particle does not move relative to frame *O*.

Fortunately, we have already derived the general Eq. (8) to analyze situations exactly like this one. Taking **A** = **r**, we may apply it to the last term of Eq. (87), to get

Transformation of velocity

$$\mathbf{v}\big|_{\text{in lab}} = \mathbf{v}_O\big|_{\text{in lab}} + (\mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}), \tag{6.88}$$

where $\boldsymbol{\omega}$ is the instantaneous angular velocity of an imaginary rigid body connected to the moving reference frame (or we may say, of the frame as such), an $\mathbf{v}$ is $d\mathbf{r}/dt$, as measured in the *moving* frame O, (Here and later in this section, all vectors without indices imply their observation from the moving frame.) Relation (88), on one hand, is a natural generalization of Eq. (10) for $\mathbf{v} \neq 0$; on the other hand, if $\boldsymbol{\omega} = 0$, it is reduced to simple Eq. (1.8) for the translational motion of frame O.

Now, in order to calculate acceleration, me may just repeat the trick: differentiate Eq. (88) over time, and then use Eq. (8) again, now for vector $\mathbf{A} = \mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}$. The result is

$$\mathbf{a}\big|_{\text{in lab}} \equiv \mathbf{a}_O\big|_{\text{in lab}} + \frac{d}{dt}(\mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}) + \boldsymbol{\omega}\times(\mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}). \tag{6.89}$$

Carrying out the differentiation in the second term, we finally get the goal equation,

Transformation of acceleration

$$\mathbf{a}\big|_{\text{in lab}} \equiv \mathbf{a}_O\big|_{\text{in lab}} + \mathbf{a} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + 2\boldsymbol{\omega}\times\mathbf{v} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}), \tag{6.90}$$

where $\mathbf{a}$ is particle's acceleration, as measured in the moving frame. Evidently, Eq. (90) is a natural generalization of the simple Eq. (1.9) to the rotating frame case.

Now let the lab frame O' be inertial; then the 2nd Newton law for a particle of mass m is

$$m\mathbf{a}\big|_{\text{in lab}} = \mathbf{F}, \tag{6.91}$$

where $\mathbf{F}$ is the vector sum of all forces action on the particle. This is simple and clear; however, in many cases it is much more convenient to work in a non-inertial reference frames. For example, describing most phenomena on Earth's surface, in is rather inconvenient to use a reference frame resting on the Sun (or in the galactic center, etc.). In order to understand what we should pay for the convenience of using the moving frame, we may combine Eqs. (90) and (91) to write

2nd Newton law in non-inertial reference frame

$$m\mathbf{a} = \mathbf{F} - m\mathbf{a}_O\big|_{\text{in lab}} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - 2m\boldsymbol{\omega}\times\mathbf{v} - m\dot{\boldsymbol{\omega}}\times\mathbf{r}. \tag{6.92}$$

This result may be interpreted in the following way: if we want to use the 2nd Newton law's analog in a non-inertial reference frame, we have to add, to the real net force $\mathbf{F}$ acting on a particle, four *pseudo-force* terms, called *inertial forces*, all proportional to particle's mass. Let us analyze them, while always remembering that these are just mathematical terms, not real forces. (In particular, it would be futile to seek for the 3rd Newton law's counterpart for an inertial force.)

The first term, $-m\mathbf{a}_O|_{\text{in lab}}$, is the only one not related to rotation, and is well known from the undergraduate mechanics. (Let me hope the reader remembers all these weight-in-the-moving-elevator problems.) Despite its simplicity, this term has subtle and interesting consequences. As an example, let us consider a planet, such as our Earth, orbiting a star and also rotating about its own axis – see Fig. 13. The bulk-distributed gravity forces, acting on a planet from its star, are not quite uniform, because they obey the $1/r^2$ gravity law (1.16a), and hence are equivalent to a single force applied to a point A slightly offset from the planet's center of mass O toward the star. For a spherically-symmetric planet, points O and A would be exactly aligned with the direction toward the star. However, real planets are not absolutely rigid, so that, due to the centrifugal "force" (to be discussed shortly), their rotation about their own axis makes them slightly elliptic – see Fig. 13. (For our Earth, this *equatorial bulge* is about 10 km in each direction.) As a result, the net gravity force does create a small torque relative to the center of mass O. On the other hand, repeating all the arguments of this section for a body (rather than a point), we may see that, in the reference frame moving with the planet, the inertial "force" $-M\mathbf{a}_O$ (which is of course equal to the total gravity force and directed from the star) is applied exactly to the center of mass and does not create a torque. As a result, this pair of forces creates a torque $\boldsymbol{\tau}$ perpendicular to both the direction toward the star and the vector connecting points *O* and *A*. (In Fig. 13, the torque vector is perpendicular to the plane of drawing). If angle δ between the planet's "polar" axis of rotation and the direction towards the star was fixed, then, as we have seen in the previous section, this torque would induce a slow axis precession about that direction. However, as a result of orbital motion, angle δ oscillates in time much faster (once a year) between values ($\pi/2 + \varepsilon$) and ($\pi/2 - \varepsilon$), where ε is the axis tilt, i.e. angle between the polar axis (direction of vectors $\mathbf{L}$ and $\boldsymbol{\omega}_{rot}$) and the normal to the *ecliptic plane* of the planet's orbit. (For the Earth, $\varepsilon \approx 23.4°$.) A straightforward averaging over these fast oscillations[16] shows that the torque leads to the polar axis precession about the axis *perpendicular* to the ecliptic plane, keeping angle ε constant. For the Earth, the period, $T_{pre} = 2\pi/\omega_{pre}$, of this *precession of the equinoxes* (or "precession of the equator"), corrected to the substantial effect of Moon's gravity, is close to 26,000 years.

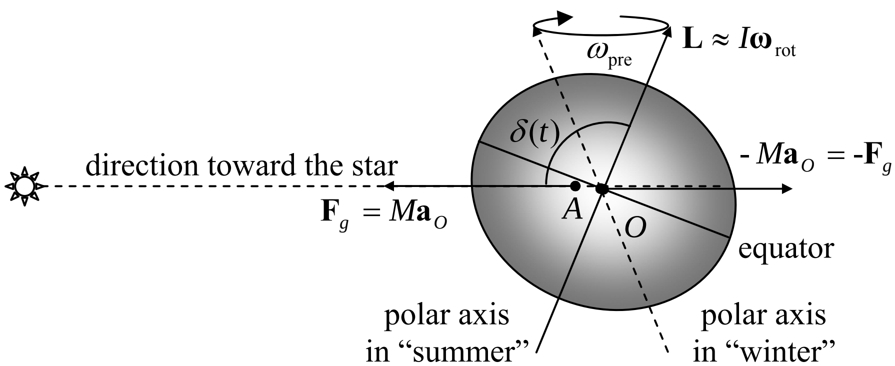

Fig. 6.13. Axial precession of a planet (with the equatorial bulge and the force line offset strongly exaggerated).

Returning to Eq. (92), the direction of the second term of its right-hand part, $\mathbf{F}_c = -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$, called the *centrifugal force*, is always perpendicular to, and directed out of the instantaneous rotation axis – see Fig. 14.

Centrifugal force

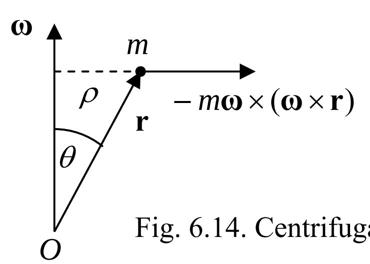

Fig. 6.14. Centrifugal "force".

Indeed, vector $\boldsymbol{\omega}\times\mathbf{r}$ is perpendicular to both $\boldsymbol{\omega}$ and $\mathbf{r}$ (in Fig. 14, normal to the picture plane and directed from the reader) and has magnitude $\omega r\sin\theta = \omega\rho$, where ρ is the distance of the particle from the rotation axis. Hence the outer vector product, with the account of the minus sign, is normal to the rotation axis $\boldsymbol{\omega}$, directed out from the axis, and equal to $\omega^2 r\sin\theta = \omega^2\rho$. The "centrifugal force" is of

[16] Details of this calculation may be found, e.g., in Sec. 5.8 of the textbook by H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics*, 3rd ed., Addison Wesley, 2002.

course just the result of the fact that the centripetal acceleration $\omega^2\rho$, explicit in the inertial reference frame, disappears in the rotating frame. For a typical location of the Earth ($\rho \sim R_E \approx 6\times10^6$ m), with its angular velocity $\omega_E \approx 10^{-4}$ s^{-1}, the acceleration is rather considerable, of the order of 3 cm/s^2, i.e. ~0.003 g, and is responsible, in particular, for the largest part of the equatorial bulge mentioned above.

As an example of using the centrifugal "force" concept, let us return again to our "testbed" problem on the bead sliding along a rotating ring – see Figs. 1.5 and 2.1. In the non-inertial reference frame attached to the ring, we have to add, to real forces $m\mathbf{g}$ and $\mathbf{N}$ acting on the bead, the horizontal centrifugal "force"[17] directed out of the rotation axis, with magnitude $m\omega^2\rho$. In the notations of Fig. 2.1, its component tangential to the ring equals $m\omega^2\rho\cos\theta = m\omega^2R\sin\theta\cos\theta$, and hence the Cartesian component of Eq. (92) along this direction is

$$ma = -mg\sin\theta + m\omega^2R\sin\theta\cos\theta. \tag{6.93}$$

With $a = R\ddot{\theta}$, this gives us the equation of motion equivalent to Eq. (2.25), which had been derived in Sec. 2.2 (in the inertial frame) using the Lagrangian formalism.

The third term in the right-hand part of Eq. (92) is the so-called *Coriolis force*,[18] which exists only if the particle moves in the rotating reference frame. Its physical sense may be understood by considering a projectile fired horizontally, say from the North Pole. From the point of view of the Earth-based observer, it will a subject of an additional Coriolis force $\mathbf{F}_C = -2m\boldsymbol{\omega}\times\mathbf{v}$, directed westward, with magnitude $2m\omega_E v$, where $\mathbf{v}$ is the main, southward component of the velocity. This force would cause the westward acceleration $a = 2\omega_E v$, and the resulting eastward deviation growing with time as $d = at^2/2 = \omega_E vt^2$ – see Fig. 15. (This formula is exact only if d is much smaller than the distance $r = vt$ passed by the projectile.) On the other hand, from the point of view of the inertial-frame observer, the projectile trajectory in the horizontal plane is a straight line, but during the flight time t, the Earth surface slips eastward from under the trajectory by distance $d = r\varphi = (vt)(\omega_E t) = \omega_E vt^2$ where $\varphi = \omega_E t$ is the azimuthal angle of the Earth rotation during the flight). Thus, both approaches give the same result. Coriolis force

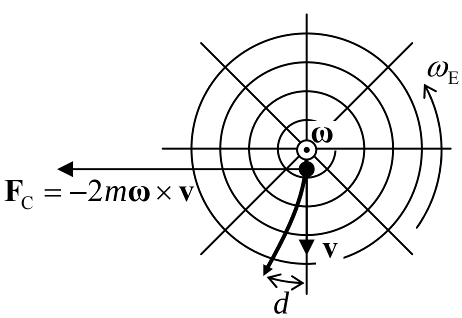

Fig. 6.15. Trajectory of a projectile fired horizontally from the North Pole, from the point of view of an Earth-bound observer looking down. Circles show parallels, straight lines mark meridians.

Hence, the Coriolis "force" is just a fancy (but frequently very convenient) way of description of a purely geometric effect pertinent to rotation, from the point of view of the observer participating in it. This force is responsible, in particular, for the higher right banks of rivers in the Northern hemisphere, regardless of the direction of their flow – see Fig. 16. Despite the smallness of the Coriolis force (for a

[17] For this problem, all other inertial "forces", besides the Coriolis force (see below) vanish, while the latter force is directed perpendicular to the ring and does not affect the bead's motion along it.
[18] Named after G.-G. Coriolis (1792-1843), who is also credited for the first unambiguous definitions of mechanical work and kinetic energy.

typical velocity of the water in a river, $v \sim 1$ m/s, it is equivalent to acceleration $a_C \sim 10^{-2}$ cm/s^2 $\sim 10^{-5}$ g), its multi-century effects may be rather prominent.[19]

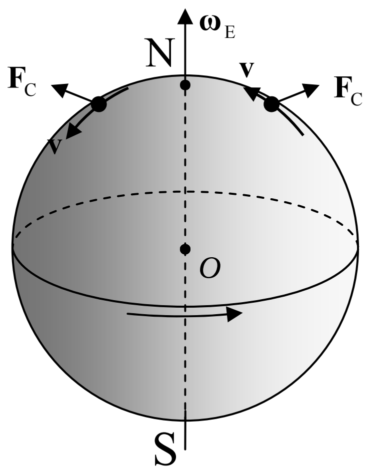

Fig. 6.16. Coriolis "forces" due to Earth's rotation, in the Northern hemisphere.

The last, fourth term of Eq. (92), $-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$, exists only when the rotation frequency changes in time, and may be interpreted as a local-position-specific addition to the first term.

Equation (92), derived above from the Newton equation (91), may be alternatively obtained from the Lagrangian approach, which also gives some important insights on energy at rotation. Let us use Eq. (88) to present the kinetic energy of the particle in an *inertial* frame in terms of **v** and **r** measured in a *rotating* frame:

$$T = \frac{m}{2}\left[\mathbf{v}_O\big|_{\text{in lab}} + (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r})\right]^2, \tag{6.94}$$

and use this expression to calculate the Lagrangian function. For the relatively simple case of particle motion in the field of potential forces, measured from a reference frame that performs pure rotation (so that $\mathbf{v}_O|_{\text{in lab}} = 0$) with a constant angular velocity $\boldsymbol{\omega}$, the result is

$$L \equiv T - U = \frac{m}{2}v^2 + m\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r}) + \frac{m}{2}(\boldsymbol{\omega}\times\mathbf{r})^2 - U = \frac{m}{2}v^2 + m\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r}) - U_{\text{ef}}, \tag{6.95}$$

where the effective potential energy,[20]

$$U_{\text{ef}} \equiv U - \frac{m}{2}(\boldsymbol{\omega}\times\mathbf{r})^2, \tag{6.96}$$

is just the sum of the real potential energy U of the particle and the so-called *centrifugal potential energy* associated with the centrifugal "inertial force":

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[19] The same force causes also the counter-clockwise circulation inside our infamous "Nor'easter" storms, in which velocity **v**, caused by lower atmospheric pressure in the middle of the cyclone, is directed toward its center.

[20] Note again the difference between the negative sign before the (always positive) second term, and the positive sign before the similar positive second term in Eq. (3.44). As was already discussed in Chapter 3, this difference hinges on different background physics: in the planetary problem, the angular momentum (and hence its component L_z) is fixed, while the corresponding angular velocity $\dot{\varphi}$ is not. On the opposite, in our current discussion, the angular velocity $\boldsymbol{\omega}$ (of the reference frame) is fixed, i.e. is independent on particle's motion.

$$\mathbf{F}_c \equiv -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) = -\nabla\left[-\frac{m}{2}(\boldsymbol{\omega}\times\mathbf{r})^2\right]. \tag{6.97}$$

Of course, the Lagrangian equations of motion derived from Eq. (95), considering the Cartesian components of **r** and **v** as generalized coordinates and velocities, coincide with Eq. (92) (with $\mathbf{a}_O|_{\text{in lab}} = \dot{\boldsymbol{\omega}} = 0$, and $\mathbf{F} = -\nabla U$), but it is very informative to have a look at a by-product of this derivation, the generalized momentum corresponding to particle's coordinate **r** as measured in the rotating reference frame,[21]

$$\boldsymbol{p} \equiv \frac{\partial L}{\partial \mathbf{v}} = m(\mathbf{v}+\boldsymbol{\omega}\times\mathbf{r}). \tag{6.98}$$

Canonical momentum at rotation

According to Eq. (88), with $\mathbf{v}_O|_{\text{in lab}} = 0$, the expression in parentheses is just $m\mathbf{v}|_{\text{in lab}}$. However, from the point of view of the moving frame, i.e. not knowing about the physical sense of vector $\boldsymbol{p} = m\mathbf{v}|_{\text{in lab}}$, we would have a reason to speak about two different momenta of the same particle, the so-called *kinetic momentum* $\mathbf{p} = m\mathbf{v}$ and the *canonical momentum* $\boldsymbol{p} = \mathbf{p} + m\boldsymbol{\omega}\times\mathbf{r}$.[22]

Now let us calculate the Hamiltonian function H and energy E as functions of the same moving-frame variables:

$$H \equiv \sum_{j=1}^{3}\frac{\partial L}{\partial v_j}v_j - L = \boldsymbol{p}\cdot\mathbf{v} - L = m\mathbf{v}\cdot(\mathbf{v}+\boldsymbol{\omega}\times\mathbf{r}) - \left[\frac{m}{2}v^2 + m\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r}) - U_{\text{ef}}\right] = \frac{mv^2}{2} + U_{\text{ef}}, \tag{6.99}$$

$$E \equiv T + U = \frac{m}{2}v^2 + m\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r}) + \frac{m}{2}(\boldsymbol{\omega}\times\mathbf{r})^2 + U = \frac{m}{2}v^2 + U_{\text{ef}} + m\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r}) + m(\boldsymbol{\omega}\times\mathbf{r})^2. \tag{6.100}$$

These expressions clearly show that E and H are *not* equal. In hindsight, this is not surprising, because the kinetic energy (94), expressed in the moving-frame variables, includes a term linear in **v,** and hence is not a quadratic-homogeneous function of this generalized velocity. The difference of these functions may be presented as

$$E - H = m\mathbf{v}\cdot(\boldsymbol{\omega}\times\mathbf{r}) + m(\boldsymbol{\omega}\times\mathbf{r})^2 = m(\mathbf{v}+\boldsymbol{\omega}\times\mathbf{r})\cdot(\boldsymbol{\omega}\times\mathbf{r}) = m\mathbf{v}\big|_{\text{in lab}}\cdot(\boldsymbol{\omega}\times\mathbf{r}). \tag{6.101}$$

Now using the operand rotation rule again, we may transform this expression into a even simpler form:[23]

$$E - H = \boldsymbol{\omega}\cdot\left(\mathbf{r}\times m\mathbf{v}\big|_{\text{in lab}}\right) = \boldsymbol{\omega}\cdot(\mathbf{r}\times\boldsymbol{p}) = \boldsymbol{\omega}\cdot\mathbf{L}\big|_{\text{in lab}}. \tag{6.102}$$

E and *H* at rotation

Let us evaluate this difference for our testbed problem – see Fig. 2.1. In this case, vector **ω** is aligned with axis z, so that of all Cartesian components of vector **L**, only component L_z is important for the scalar product (102). This component evidently equals $I_z\omega = m\rho^2\omega = m\omega R^2\sin^2\theta$, so that

$$E - H = m\omega^2R^2\sin^2\theta, \tag{6.103}$$

[21] $\partial L/\partial\mathbf{v}$ is just a shorthand for a vector with Cartesian components $\partial L/\partial v_j$. In a different language, this is the gradient of L in the velocity space.

[22] A very similar situation arises at the motion of a particle with electric charge q in magnetic field **B**. In that case the role of the additional term $\boldsymbol{p} - \mathbf{p} = m\boldsymbol{\omega}\times\mathbf{r}$ is played by product $q\mathbf{A}$, where **A** is the vector-potential of the field ($\mathbf{B} = \nabla\times\mathbf{A}$) – see, e.g., EM Sec. 9.7, and in particular Eqs. (9.183) and (9.192).

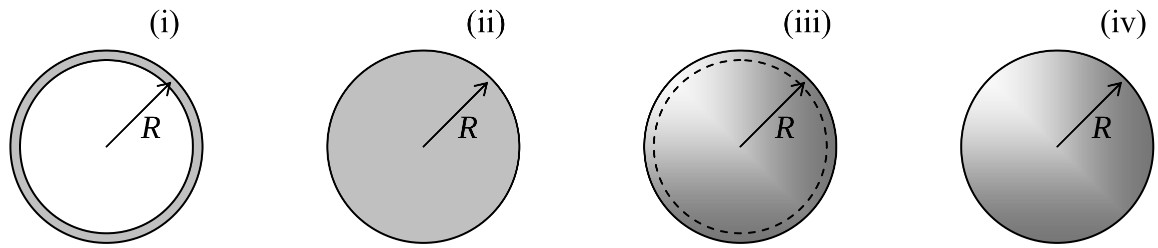
[23] Note that by definition (1.36), angular momenta **L** of particles merely add up. As a result, Eq. (102) is valid for an arbitrary system of particles.

i.e. the same result that follows from the direct subtraction of Eqs. (2.40) and (2.41).

The last form of Eq. (99) shows that in the rotating frame, the Hamiltonian function of a particle has a very simple physical sense. It is conserved, and hence may serve as an integral of motion, in many important situations when **L**, and hence E, are not – our testbed problem is again a very good example.

6.7. Exercise problems

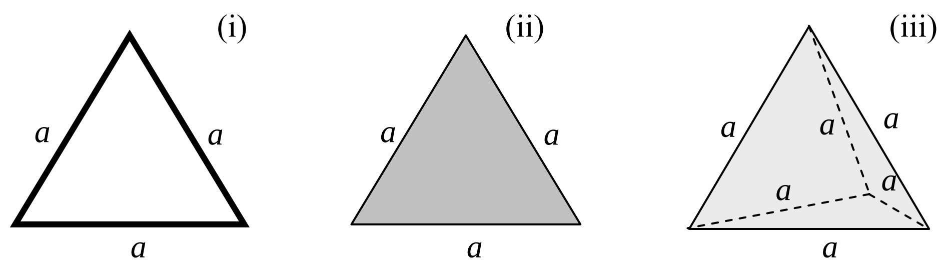
6.1. Calculate the principal moments of inertia for the following rigid bodies:



(i) a thin, plane round hoop,
(ii) a flat uniform round disk,
(iii) a thin spherical shell, and
(iv) a uniform solid sphere.

Compare the results assuming that all the bodies have the same radius R and mass M.

6.2. Calculate the principal moments of inertia for the following rigid bodies (see Fig. below):

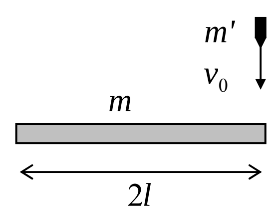


(i) an equilateral triangle made of thin rods with a uniform linear mass density μ,
(ii) a thin plate in the shape of an equilateral triangle, with a uniform areal mass density σ, and
(iii) a tetrahedral pyramid made of a heavy material with a uniform bulk mass density ρ.

Assuming that the total mass of the three bodies is the same, compare the results and give an interpretation of their difference.

6.3. Prove that Eqs. (34)-(36) are valid for rotation about a fixed axis, even if it does not pass through the center of mass, if all distances ρ_z are measured from that axis.

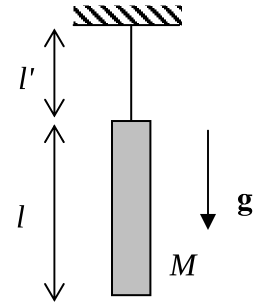
6.4. The end of a uniform, thin, heavy rod of length $2l$ and mass m, initially at rest, is hit by a bullet of mass m', flying with velocity v_0, which gets stuck in the stick - see Fig. on the right. Use two different approaches to calculate the velocity of the opposite end of the rod right after the collision.



6.5. A uniform ball is placed on a horizontal plane, while rotating with an angular velocity ω_0, but having no initial linear velocity. Calculate the angular velocity after ball's slippage stops, assuming the usual simple approximation of the kinetic friction force: $F_f = \mu N$, where N is a pressure between the surfaces, and μ is a velocity-independent coefficient.

6.6. A body may rotate about fixed horizontal axis A - see Fig. 5. Find the frequency of its small oscillations, in a uniform gravity field, as a function of distance l of the axis from body's center of mass O, and analyze the result.

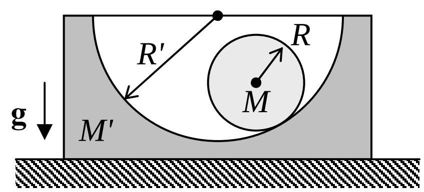
6.7. A thin uniform bar of mass M and length l is hung on a light thread of length l' (like a "chime" bell – see Fig. on the right). Find:



(i) the equations of motion of the system (within the plane of drawing);
(ii) the eigenfrequencies of small oscillations near the equilibrium;
(iii) the distribution coefficients for each oscillation mode.

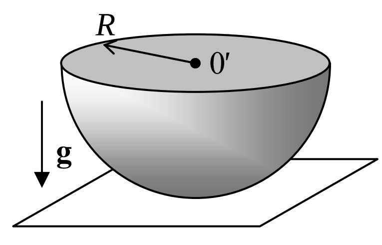
Sketch the oscillation modes for the particular case $l = l'$.

6.8. A solid, uniform, round cylinder of mass M can roll, without slipping, over a concave, round cylindrical surface of a block of mass M', in a uniform gravity field – see Fig. on the right. The block can slide without friction on a horizontal surface. Using the Lagrangian formalism,



(i) find the frequency of small oscillations of the system near the equilibrium, and
(ii) sketch the oscillation mode for the particular case $M' = M$, $R' = 2R$.

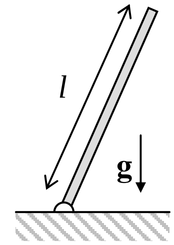
6.9. A uniform solid hemisphere of radius R is placed on a horizontal plane – see Fig. on the right. Find the frequency of its small oscillations within a vertical plane, for two ultimate cases:



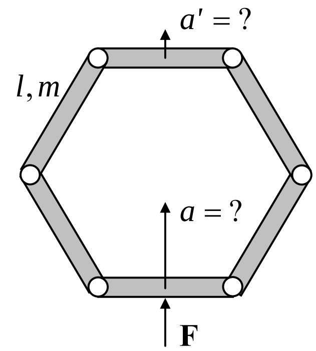
(i) there is no *friction* between the hemisphere and plane surfaces, and
(ii) the static friction is so strong that there is no *slippage* between these surfaces.

6.10. For the "sliding ladder" problem started in Sec. 3 (see Fig. 7), find the critical value α_c of angle α at which the ladder loses contact with the vertical wall, assuming that it starts sliding from the vertical position with a negligible initial velocity.

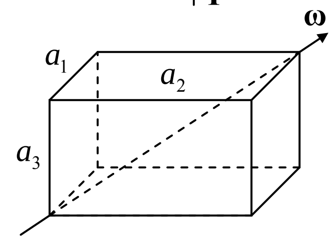
6.11.* A rigid, straight, uniform rod of length l, with the lower end on a pivot, falls in a uniform gravity field – see Fig. on the right. Neglecting friction, calculate the distribution of the bending torque τ along its length, and analyze the result.



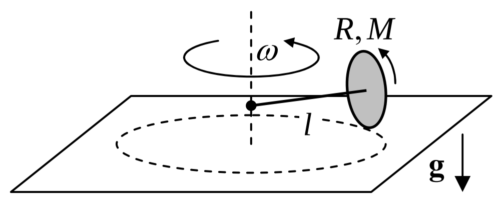
6.12. Six similar, uniform rods of length l and mass m are connected by light joints so that they may rotate, without friction, versus each other, forming a planar polygon. Initially, the polygon was at rest, and had the correct hexagon shape - see Fig. on the right. Suddenly, an external force **F** is applied to the middle of one rod, in the direction of hexagon's symmetry center. Calculate the accelerations: of the rod to which the force is applied (a), and of the opposite rod (a'), immediately after the application of the force.



6.13. A rectangular cuboid (parallelepiped) with sides a_1, a_2, and a_3, made of a material with constant density ρ, is rotated, with a constant angular velocity ω, about one of its space diagonals – see Fig. on the right. Calculate the torque $\boldsymbol{\tau}$ necessary to sustain such rotation.



6.14. One end of a light shaft of length l is firmly attached to the center of a uniform solid disk of radius $R << l$ and mass M, whole plane is perpendicular to the shaft. Another end of the shaft is attached to a vertical axis (see Fig. on the right) so that the shaft may rotate about the axis without friction. The disk rolls, without slippage, over a horizontal surface, so that the whole system rotates about the vertical axis with a constant angular velocity ω. Calculate the (vertical) supporting force exerted on the disk by the surface.



6.15. An air-filled balloon is placed inside a container filled with water that moves in space, in a negligible gravity field. Suddenly, force **F** is applied to the container, pointing in a certain direction. What direction would the balloon move relative to the container?

6.16. Calculate the height of solar tides on a large ocean, using the following simplifying assumptions: the tide period (½ of Earth's day) is much longer than the period of all ocean waves, the Earth (of mass M_E) is a sphere of radius R_E, and its distance r_S from the Sun (of mass M_S) is constant and much larger than R_E.

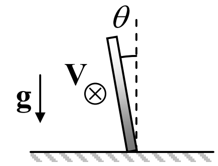
6.17.* A satellite is on a circular orbit, of radius R, around the Earth.

(i) Write the equations of motion of a small body as observed from the satellite, and simplify them for the case when body's motion is limited to a close vicinity of the satellite.

(ii) Use the equations to prove that at negligible external forces (in particular, negligible gravitational attraction to the satellite) the body may be placed on an elliptical trajectory around satellite's center of mass, within its plane of rotation about Earth. Calculate the ellipse's orientation and eccentricity.

6.18.* A non-spherical shape of an artificial satellite may ensure its stable angular orientation relative to Earth's surface, advantageous for many practical goals. Modeling the satellite as a strongly elongated, axially-symmetric body, moving around the Earth on a circular orbit of radius R, find its stable orientation, and analyze possible small oscillations of the satellite's symmetry axis around this equilibrium position.

6.19. A thin coin of radius r is rolled, with velocity V, on a horizontal surface without slippage. What should be coin's tilt angle θ (see Fig. on the right) for it to roll on a circle of radius $R >> r$? Modeling the coin as a very thin, uniform disk, and assuming that angle θ is small, solve this problem in:



(i) an inertial ("lab") reference frame, and
(ii) the non-inertial reference frame moving with coin's center of mass (but not rotating with it).

6.20. Two planets are on the circular orbit around their common center of mass. Calculate the effective potential energy of a much lighter mass (say, a spaceship) rotating with the same angular velocity, on the line connecting the planets. Sketch the plot of function the radial dependence of U_{ef} and find out the number of so-called *Lagrange points* is which the potential energy has local maxima. Calculate their position approximately in the limit when one of the planets is much more massive than the other one.

6.21. A small body is dropped down to the surface of Earth from height $h << R_E$, without initial velocity. Calculate the magnitude and direction of its deviation from the vertical, due to the Earth rotation. Estimate the effect's magnitude for a body dropped from the Empire State Building.

6.22. Use Eq. (94) to calculate the generalized momentum and derive the Lagrange equation of motion of a particle, considering L a function of **r** and **v** as measured in a non-inertial but non-rotating reference frame.