Chapter 4. Oscillations

In this course, oscillations in 1D (and effectively 1D) systems are discussed in detail, because of their key importance for physics and engineering. We will start with the so-called "linear" oscillator whose differential equation of motion is linear and hence allows the full analytical solutions, and then proceed to "nonlinear" and parametric systems whose dynamics may be only explored by either approximate analytical or numerical methods.

4.1. Free and forced oscillations

In Sec. 3.2 we briefly discussed oscillations in a very important Hamiltonian system - a 1D *harmonic oscillator* described by a simple 1D Lagrangian[1]

$$L \equiv T(\dot{q}) - U(q) = \frac{m}{2}\dot{q}^2 - \frac{\kappa}{2}q^2, \tag{4.1}$$

whose Lagrangian equation of motion,

Harmonic oscillator's equation

$$m\ddot{q} + \kappa q = 0, \quad \text{i.e. } \ddot{q} + \omega_0^2 q = 0, \quad \text{with } \omega_0^2 \equiv \frac{\kappa}{m} \geq 0, \tag{4.2}$$

is a *linear homogeneous* differential equation. Its general solution is presented by Eq. (3.16), but it is frequently useful to recast it into another, amplitude-phase form:

Harmonic oscillator's motion

$$q(t) = u\cos\omega_0 t + v\sin\omega_0 t = A\cos(\omega_0 t - \varphi), \tag{4.3a}$$

where A is the *amplitude* and φ the *phase* of the oscillations, which are determined by the initial conditions. Mathematically, it is frequently easier to work with sinusoidal functions as complex exponents, by rewriting Eq. (3a) in one more form:[2]

$$q(t) = \mathrm{Re}\left[Ae^{-i(\omega_0 t - \varphi)}\right] = \mathrm{Re}\left[ae^{-i\omega_0 t}\right], \tag{4.3b}$$

Real and complex amplitudes

where a is the *complex amplitude* of the oscillations:

$$a \equiv Ae^{i\varphi}, \quad |a| = A, \quad \mathrm{Re}\,a = A\cos\varphi = u, \quad \mathrm{Im}\,a = A\sin\varphi = v. \tag{4.4}$$

Equations (3) represent the so-called *free oscillations* of the system, that are physically due to the initial energy of the system. At an account for dissipation, i.e. energy leakage out of the system, such oscillations decay with time. The simplest model of this effect is represented by an additional *linear friction* ("drag") *force* that is proportional to the generalized velocity and directed opposite to it:

[1] For the notation simplicity, in this chapter I will drop indices "ef" in the energy components T and U, and parameters like m, κ, etc. However, the reader should still remember that T and U do not necessarily coincide with the real kinetic and potential energies (even if those energies may be uniquely identified) – see Sec. 3.1.

[2] Note that this is the so-called physics convention. Most engineering texts use the opposite sign in the imaginary exponent, $\exp\{-i\omega t\} \to \exp\{i\omega t\}$, with the corresponding sign implications for intermediate formulas, but (of course) similar final results for real variables.

$$F_v = -\eta \dot{q}\,, \tag{4.5}$$

where constant η is called the *drag coefficient*.[3] The inclusion of this force modifies the equation of motion (2) to become

$$m\ddot{q} + \eta\dot{q} + \kappa q = 0\,. \tag{4.6a}$$

This equation is frequently presented in the form

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0, \quad \text{with } \delta \equiv \frac{\eta}{2m}, \tag{4.6b}$$

Free oscillator with damping

where parameter δ is called the *damping coefficient*. Note that Eq. (6) is still a linear homogeneous second-order differential equation, and its general solution still has the form of the sum (3.13) of two exponents of the type $\exp\{\lambda t\}$, with arbitrary pre-exponential coefficients. Plugging such an exponent into Eq. (4), we get the following algebraic characteristic equation for λ:

$$\lambda^2 + 2\delta\lambda + \omega_0^2 = 0. \tag{4.7}$$

Solving this quadratic equation, we get

$$\lambda_\pm = -\delta \pm i\omega_0', \quad \text{where } \omega_0' \equiv \left(\omega_0^2 - \delta^2\right)^{1/2}, \tag{4.8}$$

so that for not very high damping ($\delta < \omega_0$)[4] we get the following generalization of Eq. (3):

$$q_{\text{free}}(t) = c_+ e^{\lambda_+ t} + c_- e^{\lambda_- t} = \left(u_0 \cos\omega_0' t + v_0 \sin\omega_0' t\right) e^{-\delta t} = A_0 e^{-\delta t}\cos\left(\omega_0' t - \varphi_0\right). \tag{4.9}$$

The result shows that, besides a certain correction to the free oscillation frequency (which is very small in the most interesting case of *low damping*, $\delta << \omega_0$), the energy dissipation leads to an exponential decay of oscillation amplitude with time constant $\tau = 1/\delta$:

$$A = A_0 e^{-t/\tau}, \quad \text{where } \tau \equiv \frac{1}{\delta} = \frac{2m}{\eta}. \tag{4.10}$$

Decaying free oscillations

A convenient, dimensionless measure of damping is the so-called *quality factor* Q (or just *Q-factor*) which is defined as $\omega_0/2\delta$, and may be rewritten in several other useful forms:

[3] Here I treat Eq. (5) as a phenomenological model, but in statistical mechanics such dissipative term may be *derived* as an average force exerted on a body by its environment whose numerous degrees of freedom are in random, though possibly thermodynamically-equilibrium states. Since such environmental force also has a random component, the dissipation is fundamentally related to *fluctuations*, and the latter effects may be neglected (as they are in this course) only if the oscillation energy is much higher than the energy scale of random fluctuations of the environment - in the thermal equilibrium at temperature T, the larger of $k_B T$ and $\hbar\omega_0/2$ - see, e.g., SM Chapter 5 and QM Chapter 7.

[4] Systems with very high damping ($\delta > \omega_0$) can hardly be called oscillators, and though they are used in engineering and physics experiment (e.g., for the shock, vibration, and sound isolation), for their discussion I have to refer the interested reader to special literature – see, e.g., C. Harris and A. Piersol, *Shock and Vibration Handbook*, 5th ed., McGraw Hill, 2002. Let me only note that at very high damping, $\delta >> \omega_0$, the system may be adequately described with just one parameter: the relaxation time $1/\lambda_+ \approx 2\delta/\omega_0^2 >> \omega_0$.

$$Q \equiv \frac{\omega_0}{2\delta} = \frac{m\omega_0}{\eta} = \frac{(m\kappa)^{1/2}}{\eta} = \pi\frac{\tau}{\mathcal{T}} = \frac{\omega_0\tau}{2}, \tag{4.11}$$

where $\mathcal{T} = 2\pi/\omega_0$ is the oscillation period in the absence of damping – see Eq. (3.29). Since the oscillation energy E is proportional to their amplitude squared, i.e. decays as $\exp\{-2t/\tau\}$, with time constant $\tau/2$, the last form of Eq. (11) may be used to rewrite the Q-factor in one more form:

$$Q = \omega_0 \frac{E}{(-\dot{E})} \equiv \omega_0 \frac{E}{\mathcal{P}}, \tag{4.12}$$

where $\mathcal{P}$ is the dissipation power. (Two other useful ways to measure Q will be discussed in a minute.) The range of Q-factors of important oscillators is very broad, all the way from $Q \sim 10$ for a human leg (with relaxed muscles), to $Q \sim 10^4$ of the quartz crystals used in "electronic" clocks and watches, all the way up to $Q \sim 10^{12}$ for microwave cavities with superconducting walls.

In contrast to the decaying free oscillations, the *forced oscillations*, induced by an external force $F(t)$, may maintain their amplitude infinitely, even at nonvanishing damping. This process may be described by a still linear but now *inhomogeneous* differential equation

Forced oscillator with damping

$$m\ddot{q} + \eta\dot{q} + \kappa q = F(t), \tag{4.13a}$$

or, more conveniently, by the following generalization of Eq. (6b):

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = f(t), \quad \text{where } f(t) \equiv F(t)/m. \tag{4.13b}$$

For a particle of mass m, confined to a straight line, Eq. (12a) is just an expression of the 2nd Newton law (or rather one of its Cartesian component). More generally, according to Eq. (1.41), Eq. (13) is valid for any dissipative 1D system whose Gibbs potential energy (1.39) has the form $U_G(q, t) = \kappa q^2/2 - F(t)q$.

The forced-oscillation solutions may be analyzed by two mathematically equivalent methods whose relative convenience depends on the character of function $f(t)$.

(i) Frequency domain. Let us present function $f(t)$ as a Fourier sum of sinusoidal harmonics:[5]

$$f(t) = \sum_{\omega} f_\omega e^{-i\omega t}. \tag{4.14}$$

Then, due to linearity of Eq. (13), its general solution may be presented as a sum of the decaying free oscillations (9) with frequency ω_0', independent of function $F(t)$, and forced oscillations due to each of the Fourier components of the force:[6]

General solution of Eq. (13)

$$q(t) = q_{\text{free}}(t) + q_{\text{forced}}(t), \quad q_{\text{forced}}(t) = \sum_{\omega} a_\omega e^{-i\omega t}. \tag{4.15}$$

Plugging Eq. (15) into Eq. (13), and requiring the factors before each $e^{-i\omega t}$ in both parts to be equal, we get

[5] Operator Re, used in Eq. (3), may be dropped here, because for any physical (real) force, the imaginary components of the sum compensate each other. This imposes the following condition on the complex Fourier amplitudes: $f_{-\omega} = f_\omega^*$, where the star means the complex conjugation.

[6] In physics, this mathematical property of linear equations is frequently called the *linear superposition principle*.

$$a_\omega = f_\omega \chi(\omega), \tag{4.16}$$

where complex function $\chi(\omega)$, in our particular case equal to

$$\chi(\omega) = \frac{1}{\left(\omega_0^2 - \omega^2\right) - 2i\omega\delta}, \tag{4.17}$$

is called either the *response function* or (especially for non-mechanical oscillators) the *generalized susceptibility*. From here, the real amplitude of oscillations under the effect of a sinusoidal force that may be represented by just one Fourier harmonic of the sum (15), is

$$A_\omega \equiv |a_\omega| = |f_\omega||\chi(\omega)|, \quad \text{with } |\chi(\omega)| = \frac{1}{\left[(\omega_0^2 - \omega^2)^2 + (2\omega\delta)^2\right]^{1/2}}. \tag{4.18}$$

Forced oscillation's amplitude

This formula describes, in particular, an increase of the oscillation amplitude A_ω at $\omega \to \omega_0$ - see Fig. 1. According to Eqs. (11) and (20), at the exact resonance,

$$|\chi(\omega)|_{\omega=\omega_0} = \frac{1}{2\omega_0\delta}, \tag{4.19}$$

so that, according to Eq. (11), the ratio of the oscillator response magnitudes at $\omega = \omega_0$ and at $\omega = 0$ ($|\chi(\omega)|_{\omega=0} = 1/\omega_0^2$) is exactly equal to the Q-factor. Thus, the response increase is especially strong in the low damping limit ($\delta << \omega_0$, i.e. $Q >> 1$); moreover at $Q \to \infty$ and $\omega \to \omega_0$ the response diverges. (This fact is very useful for the approximate methods to be discussed later in this chapter.) This is of course the classical description of the famous phenomenon of *resonance*, so ubiquitous in physics.

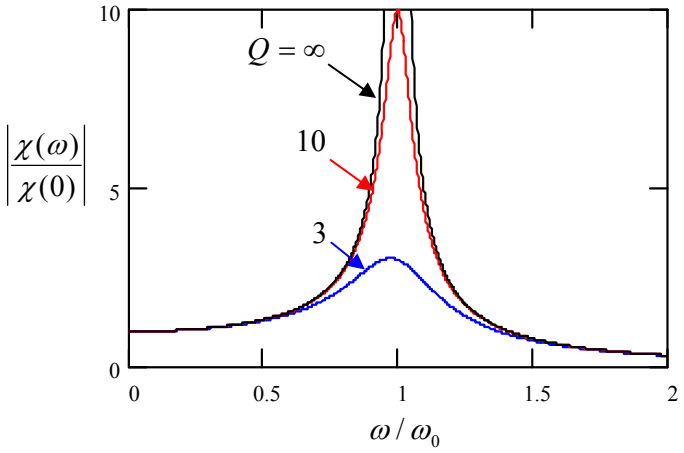

Fig. 4.1. Resonance in a harmonic oscillator (13), for several values of the Q-factor.

Due to the increase of the resonance peak height, its width is inversely proportional to Q. Quantitatively, in the most interesting low-damping limit, $Q >> 1$, the reciprocal Q-factor gives the normalized value of the so-called FWHM ("full-width at half-maximum") of the resonance curve:

$$\frac{\Delta\omega}{\omega_0} = \frac{1}{Q}. \tag{4.20}$$

Indeed, $\Delta\omega$ is defined as the difference ($\omega_+ - \omega_-$) between the two values of ω at that the square of oscillator response function, $|\chi(\omega)|^2$ (which is, in particular, proportional to the oscillation energy),

equals a half of its resonance value (19). In the low damping limit, both these points are very close to ω_0, so that in the first (linear) approximation in $(\omega - \omega_0) << \omega_0$, ω we can take $(\omega_0^2 - \omega^2) \equiv -(\omega+\omega_0)(\omega-\omega_0) \approx (-2\omega\xi) \approx (-2\omega_0\xi)$, where

$$\xi \equiv \omega - \omega_0 \tag{4.21}$$

is a convenient parameter called *detuning*. (We will repeatedly use it later in this chapter.) In this approximation, the second of Eqs. (18) is reduced to

$$\left|\chi(\omega)\right|^2 = \frac{1}{4\omega^2\left(\delta^2 + \xi^2\right)}. \tag{4.22}$$

As a result, points $\omega_\pm$ correspond to $\xi^2 = \delta^2$, i.e. $\omega_\pm = \omega_0 \pm \delta = \omega_0(1 \pm 1/2Q)$, so that $\Delta\omega \equiv \omega_+ - \omega_- = \omega_0/Q$, thus proving Eq. (20).

(ii) Time domain. Returning to the general problem of linear oscillations, one may argue that Eqs. (9), (15)-(17) provide a full solution of the forced oscillation problem. This is formally correct, but this solution may be very inconvenient if the external force is far from sinusoidal function of time. In this case, we should first calculate the complex amplitudes f_ω participating in the Fourier sum (14). In the general case of non-periodic $f(t)$, this is actually the Fourier integral,

$$f(t) = \int_{-\infty}^{+\infty} f_\omega e^{-i\omega t}\, dt, \tag{4.23}$$

so that f_ω should be calculated using the reciprocal Fourier transform,

$$f_\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t')e^{i\omega t'}\, dt'. \tag{4.24}$$

Now we can use Eq. (16) for each Fourier component of the resulting forced oscillations, and rewrite the last of Eqs. (15) as

$$\begin{aligned} q_{\text{forced}}(t) &= \int_{-\infty}^{+\infty} a_\omega e^{-i\omega t}\, d\omega = \int_{-\infty}^{+\infty} \chi(\omega) f_\omega e^{-i\omega t}\, d\omega = \int_{-\infty}^{+\infty} d\omega\, \chi(\omega)\frac{1}{2\pi}\int_{-\infty}^{+\infty} dt'\, f(t')e^{i\omega(t'-t)} \\ &= \int_{-\infty}^{+\infty} dt'\, f(t')\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} d\omega\, \chi(\omega)e^{i\omega(t'-t)}\right], \end{aligned} \tag{4.25}$$

with the response function $\chi(\omega)$ given, in our case, by Eq. (17). Besides requiring two integrations, Eq. (25) is conceptually uncomforting: it seems to indicate that the oscillator's coordinate at time t depends not only on the external force exerted at earlier times $t' < t$, but also in future times. This would contradict one of the most fundamental principles of physics (and indeed, science as a whole), the *causality*: no effect may precede its cause.

Fortunately, a straightforward calculation (left for reader's exercise) shows that the response function (17) satisfies the following rule:[7]

[7] This is true for all systems in which $f(t)$ represents a cause, and $q(t)$ its effect. Following tradition, I discuss the frequency-domain expression of this causality relation (called the *Kramers-Kronig relations*) in the *Classical Electrodynamics* part of this lecture series – see EM Sec. 7.3.

$$\int_{-\infty}^{+\infty} \chi(\omega) e^{-i\omega\tau} d\omega = 0, \qquad \text{for } \tau < 0. \tag{4.26}$$

This fact allows the last form of Eq. (25) to be rewritten in either of the following equivalent forms:

$$q_{\text{forced}}(t) = \int_{-\infty}^{t} f(t') G(t-t') dt' = \int_{0}^{\infty} f(t-\tau) G(\tau) d\tau, \tag{4.27}$$

Linear system's response

where $G(\tau)$, defined as the Fourier transform of the response function,

$$G(\tau) \equiv \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(\omega) e^{-i\omega\tau} d\omega, \tag{4.28}$$

Temporal Green's function

is called the (*temporal*) *Green's function* of the system. According to Eq. (26), $G(\tau) = 0$ for all $\tau < 0$.

While the second form of Eq. (27) is more convenient for calculations, its first form is more clear conceptually. Namely, it expresses the linear superposition principle in time domain, and may be interpreted as follows: the full effect of force $f(t)$ on an oscillator (actually, any *linear system*[8]) may be described as a sum of effects of short pulses of duration dt' and magnitude $f(t')$:

$$q_{\text{forced}}(t) = \lim_{\Delta t' \to 0} \sum_{t'=-\infty}^{t} G(t-t') f(t') \Delta t'. \tag{4.29}$$

- see Fig. 2. The Green's function $G(\tau)$ thus describes the oscillator response to a unit pulse of force, measured at time $\tau = t - t'$ after the pulse.

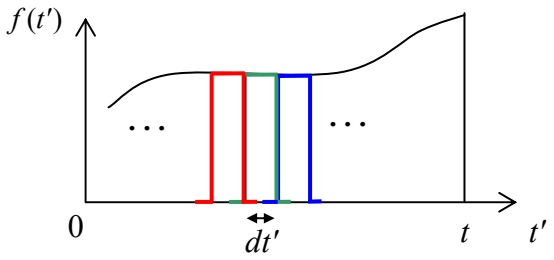

Fig. 4.2. Presentation of the force as a function of time as a sum of short pulses.

Mathematically, it is more convenient to go to the limit $dt' \to 0$ and describe the elementary, unit-area pulse by Dirac's δ-function,[9] thus returning to Eq. (27). This line of reasoning also gives a convenient way to calculate the Green's function. Indeed, for the particular case,

$$f(t) = \delta(t - t_0), \quad \text{with } t_0 < t, \tag{4.30}$$

Eq. (27) yields $q(t) = G(t - t_0)$. In particular, if $t > 0$, we may take $t_0 = 0$; then $q(t) = G(t)$. Hence the Green's function may be calculated as a solution of the differential equation of motion of the system, in our case, Eq. (13), with the δ-functional right-hand part:

$$\frac{d^2 G(\tau)}{d\tau^2} + 2\delta \frac{dG(\tau)}{d\tau} + \omega_0^2 G(\tau) = \delta(\tau), \tag{4.31}$$

[8] This is a very unfortunate, but common jargon, meaning "the system described by linear equations of motion".
[9] For a reminder of the basic properties of the δ-function, see MA Sec. 14.

and zero initial conditions:

$$G(-0) = \frac{dG}{d\tau}(-0) = 0, \tag{4.32}$$

where $t = -0$ means the instant immediately preceding $t = 0$.

This calculation may be simplified even further. Let us integrate both sides of Eq. (31) over a infinitesimal interval including the origin, e.g. $[-d\tau/2, +d\tau/2]$, and then follow the limit $d\tau \to 0$. Since Green's function has to be continuous because of its physical sense as the (generalized) coordinate, all terms in the left hand part but the first one vanish, while the first term yields $dG/d\tau\big|_{+0} - dG/d\tau\big|_{-0}$. Due to the second of Eqs. (32), the last of these two terms equals zero, while the right-hand part yields 1. Thus, $G(\tau)$ may be calculated for $\tau > 0$ (i.e. for all times when $G(\tau) \neq 0$) by solving the *homogeneous* version of system's equation of motion for $\tau > 0$, with the following special initial conditions:

$$G(0) = 0, \quad \frac{dG}{d\tau}(0) = 1. \tag{4.33}$$

This approach gives us a convenient way for calculation of Green's functions of linear systems. In particular for the oscillator with not very low damping ($\delta > \omega_0$, i.e. $Q > ½$), imposing boundary conditions (33) on the general free-oscillation solution (9), we immediately get[10]

Oscillator's Green's function

$$G(\tau) = \frac{1}{\omega_0'} e^{-\delta\tau} \sin \omega_0' \tau\,. \tag{4.34}$$

Equations (27) and (34) provide a very convenient recipe for solving most forced oscillations problems. As a very simple example, let us calculate the transient process in an oscillator under the effect of a constant force being turned on at $t = 0$:

$$f(t) = \begin{cases} 0, & t < 0, \\ f_0, & t > 0, \end{cases} \tag{4.35}$$

provided that at $t < 0$ the oscillator was at rest, so that $q_{\text{free}}(t) \equiv 0$. Then the second form of Eq. (27) yields

$$q(t) = \int_0^\infty f(t-\tau) G(\tau)\, d\tau = f_0 \int_0^t \frac{1}{\omega_0'} e^{-\delta\tau} \sin \omega_0' \tau \; d\tau. \tag{4.36}$$

The simplest way to work out such integrals is to present the sine function as the imaginary part of $\exp\{i\omega_0' t\}$, and merge the two exponents, getting

$$q(t) = f_0 \frac{1}{\omega_0'} \operatorname{Im}\left[\frac{1}{\delta + i\omega_0'} e^{-\delta\tau - i\omega_0'\tau} \right]_0^t = \frac{F_0}{k}\left[1 - e^{-\delta t}\left(\cos \omega_0' t + \frac{\delta}{\omega_0'} \sin \omega_0' t \right) \right]. \tag{4.37}$$

This result, plotted in Fig. 3, is rather natural: it describes nothing more than the transient from the initial equilibrium position $q = 0$ to the new equilibrium position $q_0 = f_0/\omega_0^2 = F_0/\kappa$, accompanied by

[10] The same result may be obtained from Eq. (28) with the response function $\chi(\omega)$ given by Eq. (19). This, more cumbersome, way is left for reader's exercise.

decaying oscillations. For this particular simple function $f(t)$, the same result might be also obtained by introducing a new variable $\tilde{q}(t) \equiv q(t) - q_0$ and solving the resulting *homogeneous* equation for $\tilde{q}$ (with appropriate initial condition $\tilde{q}(0) = -q_0$), but for more complicated functions $f(t)$ the Green's function approach is irreplaceable.

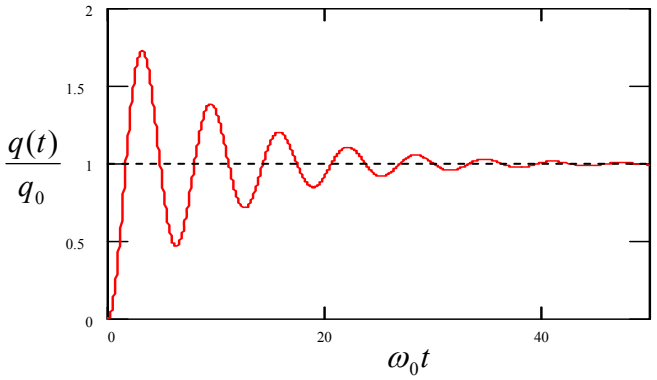

Fig. 4.3. Transient process in a linear oscillator, induced by a step-like force $f(t)$, for the particular case $\delta/\omega_0 = 0.1$ (i.e., $Q = 5$).

Note that for any particular linear system, its Green's function should be calculated only once, and then may be repeatedly used in Eq. (27) to calculate the system response to various external forces - either analytically or numerically. This property makes the Green's function approach very popular in many other fields of physics - with the corresponding generalization or re-definition of the function.[11]

4.2. Weakly nonlinear oscillations

In comparison with systems discussed in the last section, which are described by linear differential equations with constant coefficients and thus allow a complete and exact analytical solution, oscillations in nonlinear systems generally present a complex and, generally, analytically intractable problem. Let us start a discussion of such *nonlinear oscillations*[12] from an important case that may be explored analytically. In many important 1D oscillators, higher terms in the potential expansion (3.10) cannot be neglected, but are small and may be accounted for approximately. If, in addition, damping is low (or negligible), the equation of motion may be presented as a slightly modified Eq. (13):

$$\ddot{q} + \omega^2 q = f(t, q, \dot{q}, ...), \tag{4.38}$$

Weakly nonlinear oscillator

where $\omega \approx \omega_0$ is the anticipated frequency of oscillations (whose choice is to a certain extent arbitrary – see below), and the right-hand part f is small (say, scales as some small dimensionless parameter $\varepsilon << 1$), and may be considered as a *perturbation*.

Since at $\varepsilon = 0$ this equation has the sinusoidal solution given by Eq. (3), one might naïvely think that at nonvanishing but small ε, the approximate solution to Eq. (38) should be sought in the form

$$q(t) = q^{(0)} + q^{(1)} + q^{(2)} + ..., \quad \text{where } q^{(n)} \propto \varepsilon^n, \tag{4.39}$$

Formal perturbative solution

with $q^{(0)} = A \cos(\omega_0 t - \varphi) \propto \varepsilon^0$. This is a good example of an apparently impeccable mathematical reasoning that would lead to a very inefficient procedure. Indeed, let us apply it to the problem we

[11] See, e.g., EM Sec. 2.7, and QM Sec. 2.2.

[12] Again, "nonlinear oscillations" is a generally accepted slang term for oscillations in systems described by nonlinear equations of motion.

already know the exact solution for, namely the free oscillations in a linear but damped oscillator, for this occasion assuming the damping to be very low, $\delta/\omega_0 \sim \varepsilon << 1$. The corresponding equation of motion, Eq. (6), may be presented in form (38) if we take $\omega = \omega_0$ and

$$f = -2\delta\dot{q}, \quad \delta \propto \varepsilon . \tag{4.40}$$

The naïve approach described above would allow us to find *small* corrections, of the order of δ, to the free, non-decaying oscillations $A\cos(\omega_0 t - \varphi)$. However, we already know from Eq. (9) that the main effect of damping is a gradual decrease of the free oscillation amplitude to zero, i.e. a very *large* change of the amplitude, though at low damping, $\delta << \omega_0$, this decay takes large time $t \sim \tau >> 1/\omega_0$. Hence, if we want our approximate method to be productive (i.e. to work at all time scales, in particular for forced oscillations with established, constant amplitude and phase), we need to account for the fact that the *small* right-hand part of Eq. (38) may eventually lead to *essential* changes of oscillation amplitude A (and sometimes, as we will see below, also of oscillation phase φ) at large times, because of the *slowly accumulating* effects of the small perturbation.[13]

This goal may be achieved by the account of these slow changes already in the "0th approximation", i.e. the basic part of the solution in expansion (39):

0th order RWA solution

$$q^{(0)} = A(t)\cos[\omega t - \varphi(t)], \quad \text{with } \dot{A}, \dot{\varphi} \to 0 \quad \text{at } \varepsilon \to 0 . \tag{4.41}$$

The approximate methods based on Eqs. (39) and (41) have several varieties and several names,[14] but their basic idea and the results in the most important approximation (41) are the same. Let me illustrate this approach on a particular, simple but representative example of a dissipative (but high-Q) pendulum driven by a weak sinusoidal external force with a nearly-resonant frequency:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 \sin q = f_0 \cos\omega t, \tag{4.42}$$

with $|\omega - \omega_0|$, $\delta << \omega_0$, and the force amplitude f_0 so small that $|q| << 1$ at all times. From what we know about the forced oscillations from Sec. 1, it is natural to identify ω in the left-hand part of Eq. (38) with the force frequency. Expanding $\sin q$ into the Taylor series in small q, keeping only the first two terms of this expansion, and moving all the small terms to the right-hand part, we can bring Eq. (42) to the canonical form (38):[15]

Duffing equation

$$\ddot{q} + \omega^2 q = -2\delta\dot{q} + 2\xi\omega q + \alpha q^3 + f_0 \cos\omega t \equiv f(t, q, \dot{q}) . \tag{4.43}$$

Here $\alpha = \omega_0^2/6$ in the case of the pendulum (though the calculations below will be valid for any α), and the second term in the right-hand part was obtained using the approximation already employed in Sec. 1:

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[13] The same flexible approach is necessary to approximations used in quantum mechanics. The method discussed here is close in spirit (but not identical) to the *WKB approximation* (see, e.g., QM Sec. 2.4) rather to the *perturbation theory* varieties (QM Ch. 6).

[14] In various texts, one can meet references to either the *small parameter method* or *asymptotic methods*. The list of scientists credited for the development of this method and its variations notably includes J. Poincaré, B. van der Pol, N. Krylov, N. Bogolyubov, and Yu. Mitroplolsky. Expression (41) itself is frequently called the *Rotating-Wave Approximation* - RWA. (The origin of the term will be discussed in Sec. 6 below.) In the view of the pioneering role of B. van der Pol in the development of this approach, in some older textbooks the rotating-wave approximation is called the "van der Pol method".

[15] This equation is frequently called the *Duffing equation* (or the equation of the *Duffing oscillator*), after G. Duffing who was the first one to carry out its (rather incomplete) analysis in 1918.

$(\omega^2 - \omega_0^2)q \approx 2\omega(\omega - \omega_0)q = 2\omega\xi q$, where $\xi \equiv \omega - \omega_0$ is the detuning parameter that was already used earlier – see Eq. (21).

Now, following the general recipe expressed by Eqs. (39) and (41), in the 1st approximation in $f \propto \varepsilon$,[16] we may look for the solution to Eq. (43) in the form

$$q(t) = A\cos\Psi + q^{(1)}(t), \quad \text{where } \Psi \equiv \omega t - \varphi, \qquad q^{(1)} \sim \varepsilon. \tag{4.44}$$

Let us plug this assumed solution into both parts of Eq. (43), leaving only the terms of the first order in ε. Thanks to our (smart :-) choice of ω in the left-hand part of that equation, the two zero-order terms in that part cancel each other. Moreover, since each term in the right-hand part of Eq. (43) is already of the order of ε, we may drop $q^{(1)} \propto \varepsilon$ from the substitution into that part at all, because this would give us only terms $O(\varepsilon^2)$ or higher. As a result, we get the following approximate equation:

$$\ddot{q}^{(1)} + \omega^2 q^{(1)} = f^{(0)} \equiv -2\delta\frac{d}{dt}(A\cos\Psi) + 2\xi\omega A\cos\Psi + \alpha\left(A\cos\Psi\right)^3 + f_0\cos\omega t. \tag{4.45}$$

According to Eq. (41), generally A and φ should be considered as (slow) functions of time. However, let us leave the analyses of transient process and system stability until the next section, and use Eq. (45) to find stationary oscillations in the system, that are established after the initial transient. For that limited task, we may take A = const, φ = const, so that $q^{(0)}$ presents sinusoidal oscillations of frequency ω. Sorting the terms in the right-hand part according to their time dependence,[17] we see that it has terms with frequencies ω and 3ω:

$$f^{(0)} = (2\xi\omega A + \frac{3}{4}\alpha A^3 + f_0\cos\varphi)\cos\Psi + (2\delta\omega A - f_0\sin\varphi)\sin\Psi + \frac{1}{4}\alpha A^3\cos 3\Psi. \tag{4.46}$$

Now comes the main trick of the rotating-wave approximation: mathematically, Eq. (45) may be viewed as the equation of oscillations in a linear, dissipation-free harmonic oscillator of frequency ω (not ω_0!) under the action of an external force represented by the right-hand part of the equation. In our particular case, it has three terms: two *quadrature* components at that very frequency ω, and the third one at frequency 3ω. As we know from our analysis of this problem in Sec. 1, if any of the first two components is nonvanishing, $q^{(1)}$ grows to infinity – see Eq. (19) with $\delta = 0$. At the same time, by the very structure of the rotating-wave approximation, $q^{(1)}$ has to be finite - moreover, small! The only way out of this contradiction is to require that amplitudes of both quadrature components of $f^{(0)}$ with frequency ω are equal to zero:

$$2\xi\omega A + \frac{3}{4}\alpha A^3 + f_0\cos\varphi = 0, \qquad 2\delta\omega A - f_0\sin\varphi = 0. \tag{4.47}$$

These two *harmonic balance equations* enable us to find both parameters of the forced oscillations: their amplitude A and phase φ. In particular, the phase may be readily eliminated from this

[16] For a mathematically rigorous treatment of the higher approximations, see, e.g., Yu. Mitropolsky and N. Dao, *Applied Asymptotic Methods in Nonlinear Oscillations*, Springer, 2004. A more laymen (and somewhat verbose) discussion of various oscillatory phenomena may be found in the classical text A. Andronov, A. Vitt, and S. Khaikin, *Theory of Oscillators*, Dover, 2011.

[17] Using the second of Eqs. (44), $\cos\omega t$ may be rewritten as $\cos(\Psi + \varphi) \equiv \cos\Psi\cos\varphi - \sin\Psi\sin\varphi$. Then using the trigonometric identity $\cos^3\Psi = (3/4)\cos\Psi + (1/4)\cos 3\Psi$ - see, e.g., MA Eq. (3.4) results in Eq. (46).

system (most easily, by expressing $\sin\varphi$ and $\cos\varphi$ from the corresponding equations, and then requiring the sum $\sin^2\varphi + \cos^2\varphi$ to equal 1), and the solution for amplitude A presented in the following implicit but convenient form:

$$A^2 = \frac{f_0^2}{4\omega^2}\frac{1}{\xi^2(A)+\delta^2}, \quad \text{where } \xi(A) \equiv \xi + \frac{3}{8}\frac{\alpha A^2}{\omega} = \omega - \left(\omega_0 - \frac{3}{8}\frac{\alpha A^2}{\omega}\right). \tag{4.48}$$

This expression differs from Eq. (22) for the linear resonance in the low-damping limit only by the replacement of the detuning ξ with its effective amplitude-dependent value $\xi(A)$ or, equivalently, of the eigenfrequency ω_0 of the resonator with its effective, amplitude-dependent value

$$\omega_0(A) = \omega_0 - \frac{3}{8}\frac{\alpha A^2}{\omega}. \tag{4.49}$$

The physical meaning of $\omega_0(A)$ is simple: this is just the frequency of free oscillations of amplitude A in a similar nonlinear system, but with zero damping. Indeed, for $\delta = 0$ and $f_0 = 0$ we could repeat our calculations, assuming that ω is an amplitude-dependent eigenfrequency $\omega_0(A)$, to be found. Then the second of Eqs. (47) is trivially satisfied, while the second of them gives Eq. (49).

Expression (48) allows one to draw the curves of this *nonlinear resonance* just by bending the linear resonance plots (Fig. 1) according to the so-called *skeleton curve* expressed by Eq. (49). Figure 4 shows the result of this procedure. Note that at small amplitude, $\omega(A) \to \omega_0$, and we return to the usual, "linear" resonance (22).

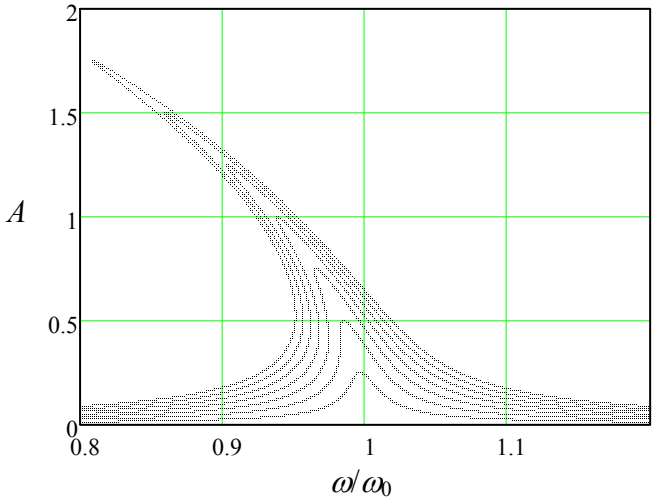


Fig. 4.4. Nonlinear resonance in the Duffing oscillator, as described by the rotating-wave approximation result (48), for the particular case $\alpha = \omega_0^2/6$, $\delta/\omega = 0.01$ (i.e. $Q = 50$), and seven values of parameter f_0/ω_0^2, increased by equal steps from 0 to 0.035.

To bring our solution to its logical completion, we should still find the first perturbation $q^{(1)}(t)$ from what is left of Eq. (45). Since the structure of this equation is similar to Eq. (13) with the force of frequency 3ω and zero damping, we may use Eqs. (16)-(17) to obtain

$$q^{(1)}(t) = -\frac{1}{32\omega^2}\alpha A^3 \cos 3(\omega t - \varphi). \tag{4.50}$$

Adding this perturbation (note the negative sign!) to the sinusoidal oscillation (41), we see that as the amplitude A of oscillations in a system with $\alpha > 0$ (e.g., a pendulum) grows, their waveform become a bit more "blunt" near the maximum deviations from the equilibrium.

Expression (50) also allows an estimate of the range of validity of the rotating-wave approximation: since it has been based on the assumption $|q^{(1)}| << |q^{(0)}| \le A$, for this particular problem we have to require $\alpha A^2/32\omega^2 << 1$. For a pendulum (with $\alpha = \omega_0^2/6$), this condition becomes $A^2 << 1/192$. Though numerical coefficients in such strong inequalities should be taken with a grain of salt, the smallness of this particular coefficient gives a good hint that the method should give very good results even for relatively large oscillations with $A \sim 1$. In Sec. 7 below, we will see that this is indeed the case.

From the mathematical viewpoint, the next step would be to calculate the next approximation

$$q(t) = A\cos\Psi + q^{(1)}(t) + q^{(2)}(t), \qquad q^{(2)} \sim \varepsilon^2, \tag{4.51}$$

and plug it into the Duffing equation (43), which (thanks to our special choice of $q^{(0)}$ and $q^{(1)}$) would retain only $\ddot{q}^{(2)} + \omega^2 q^{(2)}$ in its left-hand part. Again, requiring that amplitudes of two quadrature components of frequency ω in the right-hand part to be zero, we may get the second-order corrections to A and φ. Then we may use the remaining part of the equation to calculate $q^{(2)}$, and then go after the third-order terms, etc. However, for most purposes the sum $q^{(0)} + q^{(1)}$, and sometimes even just the crudest approximation $q^{(0)}$ alone, are completely sufficient. For example, according to Eq. (50), for a simple pendulum ($\alpha = \omega_0^2/6$) swinging as much as between the opposite horizontal positions ($A = \pi/2$), the 1st order correction $q^{(1)}$ is of the order of 0.5%. (Soon beyond this value, completely new dynamic phenomena start – see Sec. 7 below, but these phenomena cannot be covered by the rotating-wave approximation, at least in our current form.) Due to this reason, higher approximations are rarely pursued either in physics or engineering.

4.3. RWA equations

A much more important issue is the stability of solutions described by Eq. (48). Indeed, Fig. 4 shows that within a certain range of parameters, these equations give three different values for the oscillation amplitude (and phase), and it is important to understand which of these solutions are stable. Since these solutions are not the fixed points in the sense discussed in the Sec. 3.2 (each point in Fig. 4 represents a nearly-sinusoidal oscillation), their stability analysis needs a more general approach that would be valid for oscillations with amplitude and phase slowly evolving in time. This approach will also enable the analysis of non-stationary (especially the initial transient) processes that are of key importance for some dynamic systems.

First of all, let us formalize the way the harmonic balance equations, such as Eqs. (47), are obtained for the general case (38) – rather than for the particular Eq. (43) considered in the last section. After plugging in the 0th approximation (41) into the right-hand part of equation (38) we have to require the amplitudes of its both quadrature components of frequency ω to be zero. From the standard Fourier analysis we know that these requirements may be presented as

$$\boxed{\overline{f^{(0)}\sin\Psi} = 0, \quad \overline{f^{(0)}\cos\Psi} = 0,} \tag{4.52}$$

Harmonic balance equations

where symbol $\overline{...}$ means time averaging – in our current case, over the period $2\pi/\omega$ of the right-hand part of Eq. (52), with the arguments calculated in the 0th approximation:

$$f^{(0)} \equiv f(t, q^{(0)}, \dot{q}^{(0)}, ...) \equiv f\left(t, A\cos\Psi, -A\omega\sin\Psi, ...\right), \quad \text{with } \Psi = \omega t - \varphi\,. \tag{4.53}$$

Now, for a transient process the contribution of $q^{(0)}$ to left-hand part of Eq. (38) is not zero any longer, because both amplitude and phase may be slow functions of time – see Eq. (41). Let us calculate this contribution. The exact result would be

$$\ddot{q}^{(0)} + \omega^2 q^{(0)} \equiv \left(\frac{d^2}{dt^2} + \omega^2 \right) A\cos(\omega t - \varphi) = \left(\ddot{A} + 2\dot{\varphi}\omega A - \dot{\varphi}^2 A \right)\cos(\omega t - \varphi) - 2\dot{A}(\omega - \dot{\varphi})\sin(\omega t - \varphi). \tag{4.54}$$

However, in the first approximation in ε, we may neglect the second derivative of A, and also the squares and products of the first derivatives of A and φ (that are all of the second order in ε), so that Eq. (54) is reduced to

$$\ddot{q}^{(0)} + \omega^2 q^{(0)} \approx 2A\dot{\varphi}\omega\cos(\omega t - \varphi) - 2\dot{A}\omega\sin(\omega t - \varphi). \tag{4.55}$$

In the right-hand part of Eq. (52), we can neglect the time derivatives of the amplitude and phase at all, because this part is already proportional to the small parameter. Hence, in the first order in ε, Eq. (38) becomes

$$\ddot{q}^{(1)} + \omega^2 q^{(1)} = f_{\text{ef}}^{(0)} \equiv f^{(0)} - \left(2A\dot{\varphi}\omega\cos\Psi - 2\dot{A}\omega\sin\Psi \right). \tag{4.56}$$

Now, applying Eqs. (52) to function $f_{\text{ef}}^{(0)}$, and taking into account that the time averages of $\sin^2\Psi$ and $\cos^2\Psi$ are both equal to ½, while the time average of the product $\sin\Psi\cos\Psi$ vanishes, we get a pair of so-called *RWA equations* (alternatively called "the reduced equations" or sometimes "the van der Pol equations") for the time evolution of the amplitude and phase:

$$\dot{A} = -\frac{1}{\omega}\overline{f^{(0)}\sin\Psi}, \quad \dot{\varphi} = \frac{1}{\omega A}\overline{f^{(0)}\cos\Psi}. \tag{4.57a}$$

Extending the definition (4) of the complex amplitude of oscillations to their slow evolution in time, $a(t) \equiv A(t)\exp\{i\varphi(t)\}$, and differentiating this relation, we see that two equations (57a) may be also re-written in the form of either one equation for a:

Alternative forms of RWA equations

$$\dot{a} = \frac{i}{\omega}\overline{f^{(0)}e^{i(\Psi+\varphi)}} \equiv \frac{i}{\omega}\overline{f^{(0)}e^{i\omega t}}, \tag{4.57b}$$

or two equations for the real and imaginary parts of $a(t) = u(t) + iv(t)$:

$$\dot{u} = -\frac{1}{\omega}\overline{f^{(0)}\sin\omega t}, \quad \dot{v} = \frac{1}{\omega}\overline{f^{(0)}\cos\omega t}. \tag{4.57c}$$

The first-order harmonic balance equations (52) are evidently just the particular case of the RWA equations (57) for stationary oscillations ($\dot{A} = \dot{\varphi} = 0$).[18]

Superficially, the system (57a) of two coupled, first-order differential equations may look more complex than the initial, second-order differential equation (38), but actually it is usually much simpler.

[18] One may ask why cannot we stick to the just one, most compact, complex–amplitude form (57b) of the RWA equations. The main reason is that when function $f(q, \dot{q}, t)$ is nonlinear, we cannot replace its real arguments, such as $q = A\cos(\omega t - \varphi)$, with their complex-function representations like $a\exp\{-i\omega t\}$ (as could be done in the linear problems considered in Sec. 4.1), and need to use real variables, such as either $\{A, \varphi\}$ or $\{u, v\}$, anyway.

For example, let us spell them out for the easy case of free oscillations a linear oscillator with damping. For that, we may reuse the ready Eq. (46) with $\alpha = f_0 = 0$, turning Eqs. (4.57a) into

$$\dot{A} = -\frac{1}{\omega}\overline{f^{(0)}\sin\Psi} = -\frac{1}{\omega}\overline{(2\xi\omega A\cos\Psi + 2\delta\omega A\sin\Psi)\sin\Psi} = -\delta A, \tag{4.58a}$$

$$\dot{\varphi} = \frac{1}{\omega A}\overline{f^{(0)}\cos\Psi} = \frac{1}{\omega A}\overline{(2\xi\omega A\cos\Psi + 2\delta\omega A\sin\Psi)\cos\Psi} = \xi. \tag{4.58b}$$

The solution of Eq. (58a) gives us the same "envelope" law $A(t) = A(0)e^{-\delta t}$ as the exact solution (10) of the initial differential equation, while the elementary integration of Eq. (58b) yields $\varphi(t) = \xi t + \varphi(0) = \omega t - \omega_0 t + \varphi(0)$. This means that our approximate solution,

$$q^{(0)}(t) = A(t)\cos\bigl[\omega t - \varphi(t)\bigr] = A(0)e^{-\delta t}\cos\bigl[\omega_0 t - \varphi(0)\bigr], \tag{4.59}$$

agrees with the exact Eq. (9), and misses only correction (8) to the oscillation frequency, that is of the second order in δ, i.e. of the order of ε^2 – beyond the accuracy of our first approximation. It is remarkable how nicely do the RWA equations recover the proper frequency of free oscillations in this autonomous system - in which the very notion of ω is ambiguous.

The situation is different at forced oscillations. For example, for the (generally, nonlinear) Duffing oscillator described by Eq. (43) with $f_0 \neq 0$, Eqs. (57a) yield the RWA equations,

$$\dot{A} = -\delta A + \frac{f_0}{2\omega}\sin\varphi, \quad A\dot{\varphi} = \xi(A)A + \frac{f_0}{2\omega}\cos\varphi, \tag{4.60}$$

which are valid for an arbitrary function $\xi(A)$, provided that the nonlinear detuning remains much smaller than the oscillation frequency. Here (after a transient), the amplitude and phase tend to the stationary states described by Eqs. (47). This means that φ becomes a constant, so that $q^{(0)} \to A\cos(\omega t - \text{const})$, i.e. the RWA equations again automatically recover the correct frequency of the solution, in this case equal to that of the external force.

Note that each stationary oscillation regime, with certain amplitude and phase, corresponds to a fixed point of the RWA equations, so that the stability of those fixed points determine that of the oscillations. In what follows, we will carry out such an analysis for several simple systems of key importance for physics and engineering.

4.4. Self-oscillations and phase locking

The rotating-wave approximation was pioneered by B. van der Pol in the late 1920s for analysis of one more type of oscillatory motion: *self-oscillations*. Several systems, e.g., electronic rf amplifiers with positive feedback, and optical media with quantum level population inversion, provide convenient means for the compensation, and even over-compensation of the intrinsic energy losses in oscillators. Phenomenologically, this effect may be described as the change of sign of the damping coefficient δ from positive to negative. Since for small oscillations the equation of motion is still linear, we may use Eq. (9) to describe its general solution. This equation shows that at $\delta < 0$, even infinitesimal deviations from equilibrium (say, due to unavoidable fluctuations) lead to oscillations with exponentially growing amplitude. Of course, in any real system such growth cannot persist infinitely, and shall be limited by

this or that effect - e.g., in the above examples, respectively, by amplifier saturation or electron population exhaustion.

In many cases, the amplitude limitation may be described reasonably well by *nonlinear damping*:

$$2\delta\dot{q} \to 2\delta\dot{q} + \beta\dot{q}^3, \tag{4.61}$$

with $\beta > 0$. Let us analyze this phenomenon, applying the rotating-wave approximation to the corresponding homogeneous differential equation:

$$\ddot{q} + 2\delta\dot{q} + \beta\dot{q}^3 + \omega_0^2 q = 0. \tag{4.62}$$

Carrying out the dissipative and detuning terms to the right hand part as f, we can readily calculate the right-hand parts of the RWA equations (57a), getting[19]

$$\dot{A} = -\delta(A)\,A, \qquad \text{where } \delta(A) \equiv \delta + \frac{3}{8}\beta\omega^2 A^2, \tag{4.63a}$$

$$A\dot{\varphi} = \xi\,A. \tag{4.63b}$$

The second of these equations has exactly the same form as Eq. (58b) for the case of decaying oscillations and hence shows that the self-oscillations (if they happen, i.e. if $A \neq 0$) have frequency ω_0 of the oscillator itself – see Eq. (59). Equation (63a) is more interesting. If the initial damping δ is positive, it has only the trivial fixed point, $A_0 = 0$ (that describes the oscillator at rest), but if δ is negative, there is also another fixed point,

$$A_1 = \left(\frac{8|\delta|}{3\beta\omega^2}\right)^{1/2}, \tag{4.64}$$

which describes steady self-oscillations with a non-zero amplitude.

Let us apply the general approach discussed in Sec. 3.2, the linearization of equations of motion, to this RWA equation. For the trivial fixed point $A_0 = 0$, the linearization of Eq. (63a) is reduced to discarding the nonlinear term in the definition of the amplitude-dependent damping $\delta(A)$. The resulting linear equation evidently shows that the system's equilibrium point, $A = A_0 = 0$, is stable at $\delta > 0$ and unstable at $\delta < 0$. (We have already discussed this *self-excitation condition* above.) The linearization of Eq. (63a) near the non-trivial fixed point A_1 requires a bit more math: in the first order in $\widetilde{A} \equiv A - A_1 \to 0$, we get

$$\dot{\widetilde{A}} \equiv \dot{A} = -\delta(A_1 + \widetilde{A}) - \frac{3}{8}\beta\omega^2(A_1 + \widetilde{A})^3 \approx -\delta\,\widetilde{A} - \frac{3}{8}\beta\omega^2 3A_1^2\widetilde{A} = \left(-\delta + 3\delta\right)\widetilde{A} = 2\delta\widetilde{A}, \tag{4.65}$$

where Eq. (64) has been used to eliminate A_1. We see that fixed point A_1 (and hence the whole process) is stable as soon as it exists ($\delta < 0$) - similar to the situation in our "testbed problem" (Fig. 2.1).

Now let us consider another important problem: the effect of a external sinusoidal force on a self-excited oscillator. If the force is sufficiently small, its effects on the self-excitation condition and the oscillation amplitude are negligible. However, if frequency ω of such weak force is close to the

[19] For that, one needs to use the trigonometric identity $\sin^3\Psi = (3/4)\sin\Psi - (1/4)\sin 3\Psi$ - see, e.g., MA Eq. (3.4).

eigenfrequency ω_0 of the oscillator, it may lead to a very important effect of *phase-locking* (also called "synchronization"). At this effect, oscillator's frequency deviates from ω_0, and becomes exactly equal to the external force's frequency ω, within a certain range

$$-\Delta \le \omega - \omega_0 < +\Delta . \tag{4.66}$$

In order to prove this fact, and also to calculate the phase locking range width 2Δ, we may repeat the calculation of the right-hand parts of the RWA equations (57a), adding term $f_0 \cos \omega t$ to the right-hand part of Eq. (62) – cf. Eqs. (42)-(43). This addition modifies Eqs. (63) as follows:[20]

$$\dot{A} = -\delta(A)\, A + \frac{f_0}{2\omega}\sin\varphi, \tag{4.67a}$$

$$A\dot{\varphi} = \xi\, A + \frac{f_0}{2\omega}\cos\varphi. \tag{4.67b}$$

If the system is self-excited, and the external force is weak, its effect on the oscillation amplitude is small, and in the first approximation in f_0 we can take A to be constant and equal to the value A_1 given by Eq. (64). Plugging this approximation into Eq. (67b), we get a very simple equation[21]

$$\boxed{\dot{\varphi} = \xi + \Delta\cos\varphi\,,} \tag{4.68}$$

Phase locking equation

where in our current case

$$\Delta \equiv \frac{f_0}{2\omega A_1}. \tag{4.69}$$

Within the range $-|\Delta| < \xi < +|\Delta|$, Eq. (68) has two fixed points on each 2π-segment of variable φ:

$$\varphi_\pm = \pm\cos^{-1}\left(-\frac{\xi}{\Delta}\right) + 2\pi n\,. \tag{4.70}$$

It is easy to linearize Eq. (68) near each point to analyze their stability in our usual way; however, let me this case to demonstrate another convenient way to do this in 1D systems, using the so-called *phase plane* – the plot of the right-hand part of Eq. (68) as a function of φ - see Fig. 5.

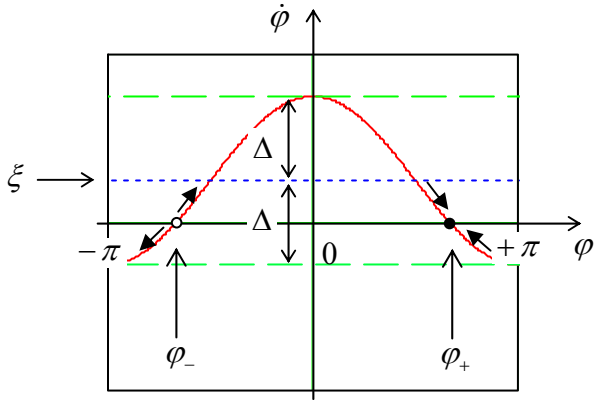


Fig. 4.5. Phase plane of a phase-locked oscillator, for the particular case $\xi = \Delta/2, f_0 > 0$.

[20] Actually, this result should be evident, even without calculations, from the comparison of Eqs. (60) and (63).

[21] This equation is ubiquitous in phase locking systems, including even some digital electronic circuits used for that purpose.

Since the positive values of this function correspond to the growth of φ in time, and vice versa, we may draw the arrows showing the direction of phase evolution. From this graphics, it is clear that one of these fixed points (for $f_0 > 0$, φ_+) is stable, while its counterpart is unstable. Hence the magnitude of Δ given by Eq. (69) is indeed the phase locking range (or rather it half) that we wanted to find. Note that the range is proportional to the amplitude of the phase locking signal - perhaps the most important feature of phase locking.

In order to complete our simple analysis, based on the assumption of fixed oscillation amplitude, we need to find the condition of validity of this assumption. For that, we may linearize Eq. (67a), for the stationary case, near value A_1, just as we have done in Eq. (65) for the transient process. The stationary result,

$$\tilde{A} \equiv A - A_1 = \frac{1}{2|\delta|}\frac{f_0}{2\omega}\sin\varphi_\pm \approx A_1\left|\frac{\Delta}{2\delta}\right|\sin\varphi_\pm , \tag{4.71}$$

shows that our assumption, $|\tilde{A}| << A_1$. and hence the final result (69), are valid if the phase locking range, 2Δ, is much smaller than $4|\delta|$.

4.5. Parametric excitation

In both problems solved in the last section, the stability analysis was easy because it could be carried out for just one slow variable, *either* amplitude *or* phase. Generally, such analysis of the RWA equations involves both these variables. The classical example of such situation is provided by one important physical phenomenon – the *parametric excitation* of oscillations. An elementary example of such oscillations is given by a pendulum with an externally-changed parameter, for example length $l(t)$ - see Fig. 6. Experiments (including those with playground swings :-) and numerical simulations show that if the length is changed (*modulated*) periodically, with frequency 2ω that is close to $2\omega_0$ and a sufficiently large swing Δl, the equilibrium position of the pendulum becomes unstable, and it starts swinging with frequency ω equal *exactly* to the half of the length modulation frequency (and hence only *approximately* equal to the average eigenfrequency ω_0 of the oscillator).

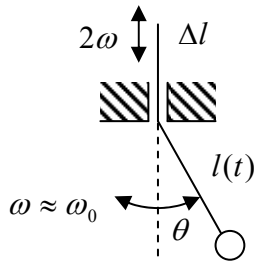


Fig. 4.6. Parametric excitation of pendulum oscillations.

For an elementary analysis of this effect we may consider the simplest case when the oscillations are small. At the lowest point ($\theta = 0$), where the pendulum moves with the highest velocity v_{max}, string's tension $\mathcal{T}$ is *higher* than mg by the centripetal force: $\mathcal{T}_{max} = mg + mv_{max}^2/l$. On the contrary, at the maximum deviation of the pendulum from the equilibrium, the force is *weakened* by string's tilt: $\mathcal{T}_{min} = mg\cos\theta_{max}$. Using the energy conservation, $E = mv_{max}^2/2 = mgl(1 - \cos\theta_{max})$, we may express these values as $\mathcal{T}_{max} = mg + 2E/l$ and $\mathcal{T}_{min} = mg - E/l$. Now, if during each oscillation period the string is pulled

up sharply and slightly by Δl ($|\Delta l| << l$) at each of its two passages through the lowest point, and is let to go down by the same amount at each of two points of the maximum deviation, the net work of the external force per period is positive:

$$W \approx 2(\mathcal{T}_{\max} - \mathcal{T}_{\min})\Delta l \approx 6\frac{\Delta l}{l}E, \tag{4.72}$$

and hence results in an increase of the oscillator's energy. If the so-called *modulation depth* $\Delta l/2l$ is sufficient, this increase may be sufficient to overcompensate the energy drained out by damping. Quantitatively, Eq. (10) shows that low damping ($\delta << \omega_0$) leads to the following energy decrease,

$$\Delta E \approx -4\pi\frac{\delta}{\omega_0}E, \tag{4.73}$$

per oscillation period. Comparing Eqs. (72) and (73), we see that the net energy flow into the oscillations is positive, $W + \Delta E > 0$, i.e. oscillation amplitude has to grow if[22]

$$\frac{\Delta l}{l} > \frac{2\pi\delta}{3\omega_0} \equiv \frac{\pi}{3Q}. \tag{4.74}$$

Since this result is independent on E, the growth of energy and amplitude is exponential (for sufficiently low E), so that Eq. (74) is the condition of parametric excitation - in this simple model.

However, this result does not account for the possible difference between the oscillation frequency ω and the eigenfrequency ω_0, and also does not clarify whether the best phase shift between the parametric oscillations and parameter modulation, assumed in the above calculation, may be sustained automatically. In order to address these issues, we may apply the rotating-wave approximation to a simple but reasonable linear equation

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2(1 + \mu\cos 2\omega t)q = 0, \tag{4.75}$$

describing the parametric excitation for a particular case of sinusoidal modulation of $\omega_0^2(t)$. Rewriting this equation in the canonical form (38),

$$\ddot{q} + \omega^2 q = f(t, q, \dot{q}) = -2\delta\dot{q} + 2\xi\omega q - \mu\omega_0^2 q\cos 2\omega t, \tag{4.76}$$

and assuming that the dimensionless ratios δ/ω and $|\xi|/\omega$, and the modulation depth μ are all much less than 1, we may use general Eqs. (57a) to get the following RWA equations:

$$\begin{aligned} \dot{A} &= -\delta A - \frac{\mu\omega}{4}A\sin 2\varphi, \\ A\dot{\varphi} &= A\xi - \frac{\mu\omega}{4}A\cos 2\varphi. \end{aligned} \tag{4.77}$$

These equations evidently have a fixed point $A_0 = 0$, but its stability analysis (though possible) is not absolutely straightforward, because phase φ of oscillations is undetermined at that point. In order to

[22] A modulation of pendulum's mass (say, by periodic pumping water in and out of a suspended bottle) gives a qualitatively similar result. Note, however, that parametric oscillations cannot be excited by modulating *any* oscillator's parameter – for example, oscillator's damping coefficient (at least if it stays positive at all times), because its does not change system's energy, just the energy drain rate.

avoid this (technical rather than conceptual) difficulty, we may use, instead of the real amplitude and phase of oscillations, either their complex amplitude $a = A\exp\{i\varphi\}$, or its Cartesian components u and v – see Eqs. (4). Indeed, for our function f, Eq. (57b) gives

$$\dot{a} = (-\delta + i\xi)a - i\frac{\mu\omega}{4}a^*, \tag{4.78}$$

while Eqs. (57c) yield

RWA equations for parametric excitation

$$\boxed{\begin{aligned} \dot{u} &= -\delta u - \xi v - \frac{\mu\omega}{4}v, \\ \dot{v} &= -\delta v + \xi u - \frac{\mu\omega}{4}u. \end{aligned}} \tag{4.79}$$

We see that in contrast to Eqs. (77), in Cartesian coordinates $\{u, v\}$ the trivial fixed point $a_0 = 0$ (i.e. $u_0 = v_0 = 0$) is absolutely regular. Moreover, equations (78)-(79) are already linear, so they do not require any additional linearization. Thus we may use the same approach as was already used in Secs. 3.2 and 4.1, i.e. look for the solution of Eqs. (79) in the exponential form $\exp\{\lambda t\}$. However, now we are dealing with two variables, and should allow them to have, for each value of λ, a certain ratio u/v. For that, we should take the partial solution in the form

$$u = c_u e^{\lambda t}, \quad v = c_v e^{\lambda t}. \tag{4.80}$$

where constants c_u and c_v are frequently called the *distribution coefficients*. Plugging this solution into Eqs. (79), we get for them the following system of two linear algebraic equations:

$$\begin{aligned} (-\delta - \lambda)c_u + (-\xi - \frac{\mu\omega}{4})c_v &= 0, \\ (\xi - \frac{\mu\omega}{4})c_u + (-\delta - \lambda)c_v &= 0. \end{aligned} \tag{4.81}$$

The characteristic equation of this system,

$$\begin{vmatrix} -\delta - \lambda & -\xi - \frac{\mu\omega}{4} \\ \xi - \frac{\mu\omega}{4} & -\delta - \lambda \end{vmatrix} \equiv \lambda^2 + 2\delta\lambda + \delta^2 + \xi^2 - \left(\frac{\mu\omega}{4}\right)^2 = 0, \tag{4.82}$$

has two roots:

$$\lambda_\pm = -\delta \pm \left[\left(\frac{\mu\omega}{4}\right)^2 - \xi^2\right]^{1/2}. \tag{4.83}$$

Requiring the fixed point to be unstable, $\mathrm{Re}\lambda_+ > 0$, we get the parametric excitation condition

$$\frac{\mu\omega}{4} > \left(\delta^2 + \xi^2\right)^{1/2}. \tag{4.84}$$

Thus the parametric excitation may indeed happen without any artificial phase adjustment: the arising oscillations self-adjust their phase to pick up energy from the external source responsible for the parameter variation.

Our key result (84) may be compared with two other calculations. First, in the case of negligible damping ($\delta = 0$), Eq. (84) turns into condition $\mu\omega/4 > |\xi|$. This result may be compared with the well-developed theory of the so-called *Mathieu equation* whose canonical form is

$$\frac{d^2y}{dv^2} + (a - 2b\cos 2v)y = 0. \tag{4.85}$$

It is evident that with the substitutions $y \to q$, $v \to \omega t$, $a \to (\omega_0/\omega)^2$, $b \to -\mu/2$, this equation is just a particular case of Eq. (75) for $\delta = 0$. In terms of Eq. (85), the result of our approximate analysis may be re-written just as $b > |a - 1|$, and is supposed to be valid for $b << 1$. This condition is shown in Fig. 7 together with the numerically calculated[23] stability boundaries of the Mathieu equation.

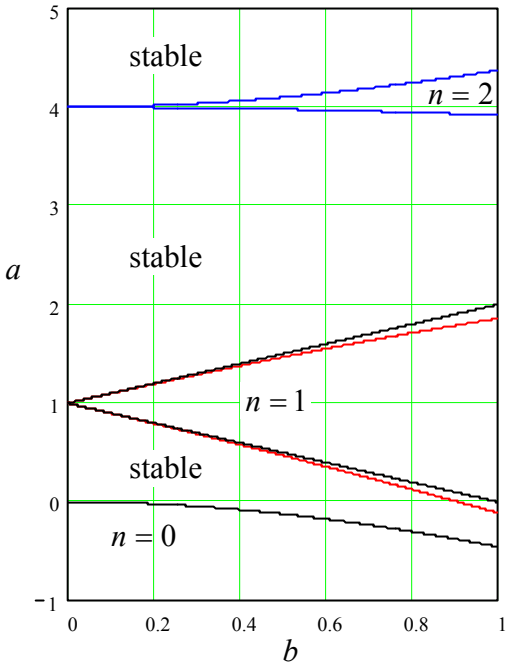


Fig. 4.7. Stability boundaries of the Mathieu equation (85), as calculated: numerically (curves) and using the rotating-wave approximation (dashed straight lines). In the regions numbered by various n the trivial solution $y = 0$ of the equation is unstable, i.e. its general solution $y(v)$ includes an exponentially growing term.

One can see that the rotating-wave approximation works just fine within its applicability limit (and beyond :-), though it fails to predict some other important features of the Mathieu equation, such as the existence of higher, more narrow regions of parametric excitation (at $a \approx n^2$, i.e. $\omega_0 \approx \omega/n$, for all integer n), and some spill-over of the stability region into the lower half-plane $a < 0$.[24] The reason of these failures is the fact that, as can be seen in Fig. 7, these phenomena do not appear in the first approximation in the parameter modulation amplitude $\mu \propto q$, that is the RWA applicability realm.

In the opposite case of finite damping but exact tuning ($\xi = 0$, $\omega \approx \omega_0$), Eq. (84) gives

$$\mu > \frac{4\delta}{\omega_0} \equiv \frac{2}{Q}. \tag{4.86}$$

[23] Such calculations may be substantially simplified by the use of the so-called *Floquet theorem*, which is also the mathematical basis for the discussion of wave propagation in periodic media – see the next chapter.

[24] This region describes, for example, the counter-intuitive stability of an inverted pendulum with the periodically modulated length, within a limited range of the modulation depth μ.

This condition may be compared with Eq. (74), taking $\Delta l/l = 2\mu$.. The comparison shows that though the structure of these conditions is similar, the numerical coefficients are different by a factor close to 2. The first reason of this difference is that the instant parameter change at optimal moments of time is more efficient then the smooth, sinusoidal variation described by (75). Even more significantly, the change of pendulum's length modulates not only its eigenfrequency ω_0, as Eq. (75) implies, but also its *mechanical impedance* $Z \equiv (g/l)^{1/2}$ – the notion to be discussed in detail in the next chapter. (Due to the time restrictions, I have to leave the analysis of the general case of the simultaneous modulation of ω_0 and Z for reader's exercise.)

Before moving on, let me summarize the most important differences between the parametric and forced oscillations:

(i) Parametric oscillations completely disappear outside of their excitation range, while the forced oscillations have a non-zero amplitude for any frequency and amplitude of the external force – see Eq. (18).

(ii) Parametric excitation may be described by a linear homogeneous equation - e.g., Eq. (75) - which cannot predict any finite oscillation amplitude within the excitation range, even at finite damping. In order to describe stationary parametric oscillations, some nonlinear effect has to be taken into account. (Again, I am leaving analyses of such effects for reader's exercises.)

One more important feature of the parametric oscillations will be discussed in the end of the next section.

4.6. Fixed point classification

RWA equations (79) give us a good pretext for a brief discussion of fixed points of a dynamic system described by *two* time-independent, *first-order* differential equations.[25] After their linearization near a fixed point, the equations for deviations can always be presented in the form similar to Eq. (79):

$$\begin{aligned} \dot{\tilde{q}}_1 &= M_{11}\tilde{q}_1 + M_{12}\tilde{q}_2, \\ \dot{\tilde{q}}_2 &= M_{21}\tilde{q}_1 + M_{22}\tilde{q}_2, \end{aligned} \tag{4.87}$$

where $M_{jj'}$ (with $j,j' = 1, 2$) are some real scalars that may be understood as elements of a 2×2 matrix M. Looking for an exponential solution of the type (80),

$$\tilde{q}_1 = c_1 e^{\lambda t}, \quad \tilde{q}_2 = c_2 e^{\lambda t}, \tag{4.88}$$

we get a more general system of two linear equations for the distribution coefficients $c_{1,2}$:

$$\begin{aligned} (M_{11} - \lambda)c_1 + M_{12}c_2 &= 0, \\ M_{21}c_1 + (M_{22} - \lambda)c_2 &= 0. \end{aligned} \tag{4.89}$$

These equations are consistent if

[25] Autonomous systems described by a *single second-order* differential equation, say $F(q, \dot{q}, \ddot{q}) = 0$, also belong to this class, because we may treat velocity $\dot{q} \equiv v$ as a new variable, and use this definition as one first-order differential equation, and the initial equation, in the form $F(q, v, \dot{v}) = 0$, as the second first-order equation.

$$\begin{vmatrix} M_{11}-\lambda & M_{12} \\ M_{21} & M_{22}-\lambda \end{vmatrix} = 0, \tag{4.90}$$

Characteristic equation of system (87)

giving us a quadratic characteristic equation

$$\lambda^2 - \lambda(M_{11}+M_{22}) + (M_{11}M_{22} - M_{12}M_{21}) = 0. \tag{4.91}$$

Its solution,[26]

$$\lambda_{\pm} = \frac{1}{2}(M_{11}+M_{22}) \pm \frac{1}{2}\left[(M_{11}-M_{22})^2 + 4M_{12}M_{21}\right]^{1/2}, \tag{4.92}$$

shows that the following situations are possible:

A. The expression under the square root, $(M_{11}- M_{22})^2 + 4M_{12}M_{21}$, is positive. In this case, both characteristic exponents $\lambda_{\pm}$ are real, and we can distinguish three sub-cases:

(i) Both λ_+ and λ_- are negative. In this case, the fixed point is evidently stable. Because of generally different magnitudes of exponents $\lambda_{\pm}$, the process presented on the phase plane $[\tilde{q}_1, \tilde{q}_2]$ (Fig. 8a) may be seen as consisting of two stages: first, a faster (with rate $|\lambda_-|$) relaxation to a linear *asymptote*,[27] and then a slower decline, with rate $|\lambda_+|$, along this line, i.e. at the virtually fixed ratio of the variables. Such fixed point is called the *stable node*.

(ii) Both λ_+ and λ_- are positive. This case (rarely met in actual physical systems) of the *unstable node* differs from the previous one only by the direction of motion along the phase plane trajectories (see dashed arrows in Fig. 8a). Here the variable ratio is also approaching a constant soon, but now the one corresponding to $\lambda_+ > \lambda_-$.

(iii) Finally, in the case of a *saddle* ($\lambda_+ > 0$, $\lambda_- < 0$) the system dynamics is different (Fig. 8b): after the rate-$|\lambda_-|$ relaxation to the λ_+-asymptote, the perturbation starts to grow, with the rate λ_+, along one of two opposite directions. (The direction is determined on which side of another straight line, called *separatrix*, the system has been initially.) It is evident that the saddle[28] is an unstable fixed point.

B. The expression under the square root, $(M_{11}- M_{22})^2 + 4\,M_{12}M_{21}$, is negative. In this case the square root in Eq. (92) is imaginary, making the real parts of both roots equal, $\mathrm{Re}\lambda_{\pm} = (M_{11} + M_{22})/2$, and their imaginary parts equal but sign-opposite. As a result, here there can be just two types of fixed points:

(i) S*table focus*, at $(M_{11} + M_{22}) < 0$. The phase plane trajectories are spirals going to the center (i.e. toward the fixed point) – see Fig. 8c with solid arrow.

(ii) *Unstable focus*, taking place at $(M_{11} + M_{22}) > 0$, differs from the stable one only by the direction of motion along the phase trajectories – see the dashed arrow in Fig. 8c.

---

[26] In terms of linear algebra, $\lambda_{\pm}$ are the *eigenvalues*, and the corresponding sets $[c_1, c_2]_{\pm}$, the *eigenvectors* of matrix M with elements $M_{jj'}$.

[27] The asymptote direction may be found by plugging the value λ_+ back into Eq. (89) and finding the corresponding ratio c_1/c_2.

[28] The term "saddle" is due to the fact that system's dynamics in this case is qualitatively similar to those of particle's motion in the 2D potential $U(\tilde{q}_1, \tilde{q}_2)$ having the shape of a horse saddle (or a mountain pass).

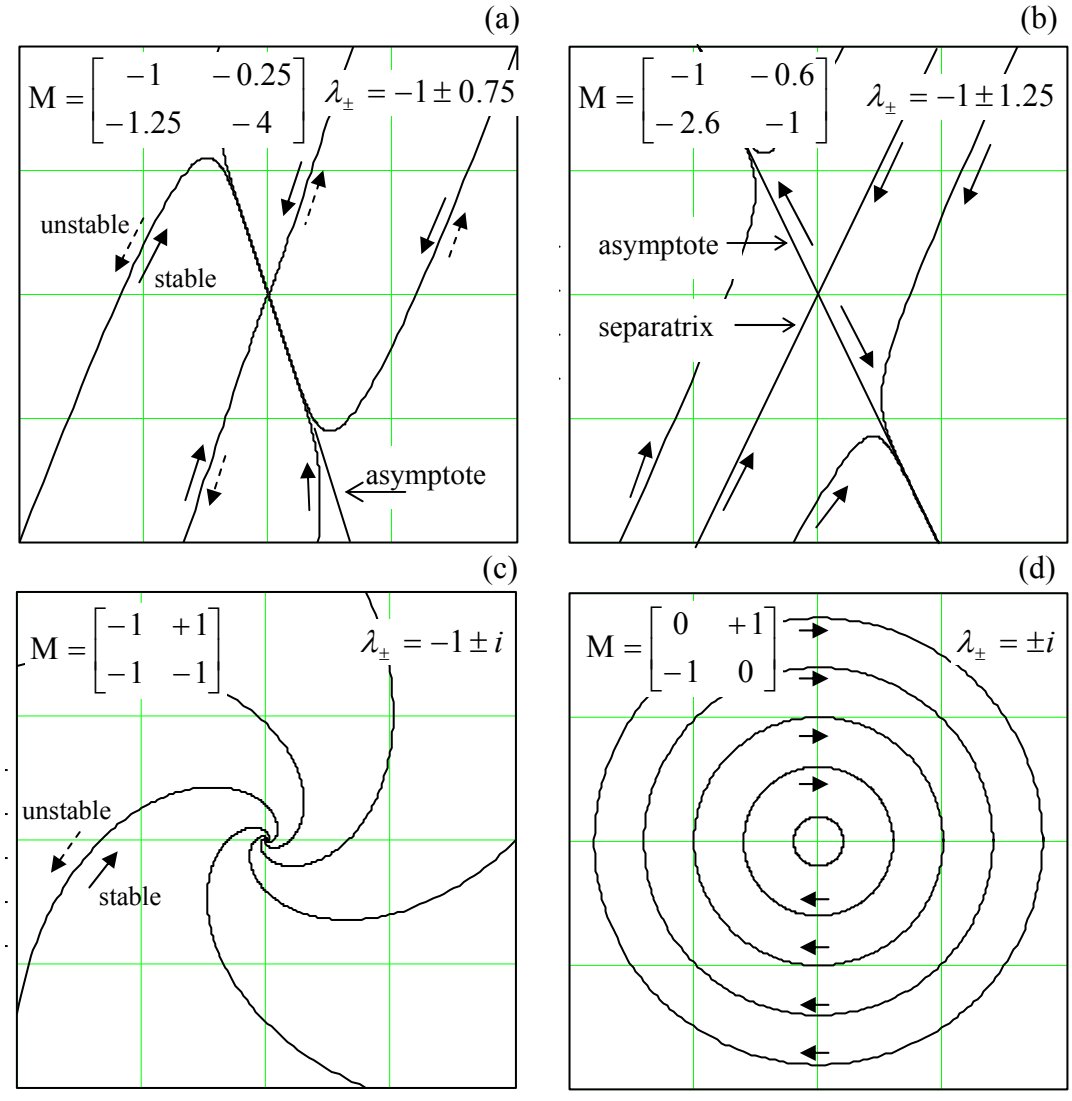


Fig. 4.8. Typical trajectories on the phase plane $[\tilde{q}_1, \tilde{q}_2]$ near fixed points of various types: (a) node, (b) saddle, (c) focus, and (d) center. The particular values of the matrix M used in the first three panels correspond to the RWA equations (81) for parametric oscillators with $\xi = \delta$, and three different values of parameter $\mu\omega/4\delta$: (a) 1.25, (b) 1.6 and (c) 0.

C. Sometimes the border case, $M_{11} + M_{22} = 0$, is also distinguished, and the corresponding fixed point is refereed to as the *center* (Fig. 8d). Considering centers a special category makes sense because such fixed points are typical for Hamiltonian systems whose first integral of motion may be frequently presented as the distance of the from a fixed point. For example, a harmonic oscillator without dissipation may be described by the system

$$\dot{q} = \frac{p}{m}, \quad \dot{p} = -m\omega_0^2 q, \tag{4.94}$$

that is evidently a particular case of Eq. (87) with $M_{11} = M_{22} = 0$, $M_{12}M_{21} = -\omega_0^2 < 0$, and hence $(M_{11}-M_{22})^2 + 4M_{12}M_{21} = -4\omega_0^2 < 0$, and $M_{11} + M_{22} = 0$. The phase plane of the system may be symmetrized by plotting q vs. the properly normalized momentum $p/m\omega_0$. On the symmetrized plane, sinusoidal oscillations of amplitude A are represented by a circle of radius A about the center-type fixed point $A = 0$. Such a circular trajectory correspond to the conservation of the oscillator's energy

$$E = \frac{m\dot{q}^2}{2} + \frac{m\omega_0^2 q^2}{2} = \frac{m\omega_0^2}{2}\left[\left(\frac{p}{m\omega_0}\right)^2 + q^2\right]. \tag{4.95}$$

This is a convenient moment for a brief discussion of the so-called *Poincaré* (or "slow-variable", or "stroboscopic") *plane*.[29] From the point of view of the rotating-wave approximation, sinusoidal oscillations $q(t) = A\cos(\omega t - \varphi)$, in particular those described by a circular trajectory on the real (or "fast") phase plane (Fig. 8c) correspond to a fixed point $\{A, \varphi\}$, which may conveniently presented by a steady geometrical point on a plane with these polar coordinates (Fig. 9a). (As follows from Eq. (4), the Cartesian coordinates on that plane are u and v.) The quasi-sinusoidal process (41), with slowly changing A and φ, may be represented by a slow motion of that point on this Poincaré plane.

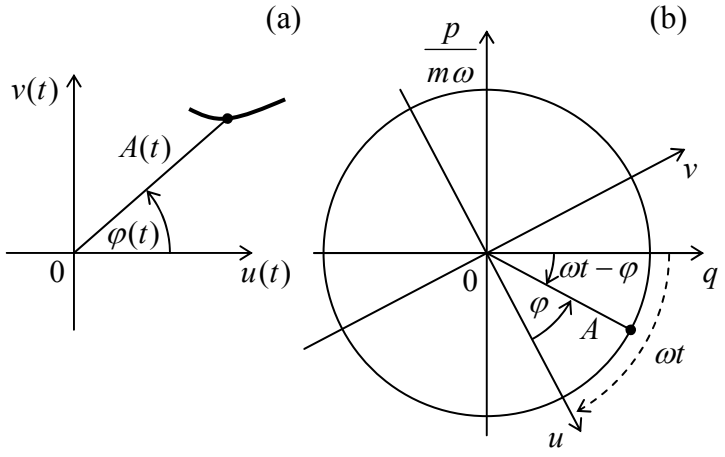


Fig. 4.9. (a) Presentation of a sinusoidal oscillation (point) and a slow transient (line) on the Poincaré plane, and (b) transfer from the "fast" phase plane to the "slow" (Poincaré) plane.

Figure 9b shows one possible way to visualize the relation between the "real" phase plane of an oscillator, with symmetrized Cartesian coordinates q and $p/m\omega_0$, and the Poincaré plane with Cartesian coordinates u and v: the latter reference frame rotates relative to the former one about the origin clockwise, with angular velocity ω.[30] Another, "stroboscopic" way to generate the Poincaré plane pattern is to have a fast glance at the "real" phase plane just once during the oscillation period $T = 2\pi/\omega$.

In many cases, such presentation is more convenient than that on the "real" phase plane. In particular, we have already seen that the RWA equations for such important phenomena as phase locking and parametric oscillations, whose original differential equations include time explicitly, are time-independent – cf., e.g., (75) and (79) describing the latter effect. This simplification brings the

[29] Named after J. H. Poincaré (1854-1912) who is credited, among many other achievements, for his contributions to special relativity (see, e.g., EM Chapter 9) and the idea of deterministic chaos (to be discussed in Chapter 9 below).

[30] This notion of phase plane rotation is the basis for the rotating-wave approximation's name. (Word "wave" has sneaked in from this method's wide application in classical and quantum optics.)

equations into the category considered in this section, and enables the classification of their fixed points, which may shed additional light on their dynamic properties.

In particular, Fig. 10 shows the classification of the trivial fixed point of a parametric oscillator, which follows from Eq. (83). As the parameter modulation depth μ is increased, the type of the trivial fixed point $A_1 = 0$ on the Poincaré plane changes from a stable focus (typical for a simple oscillator with damping) to a stable node and then to a saddle describing the parametric excitation. In the last case, the two directions of the perturbation growth, so prominently featured in Fig. 8b, correspond to the two possible values of the oscillation phase φ, with the phase choice determined by initial conditions.

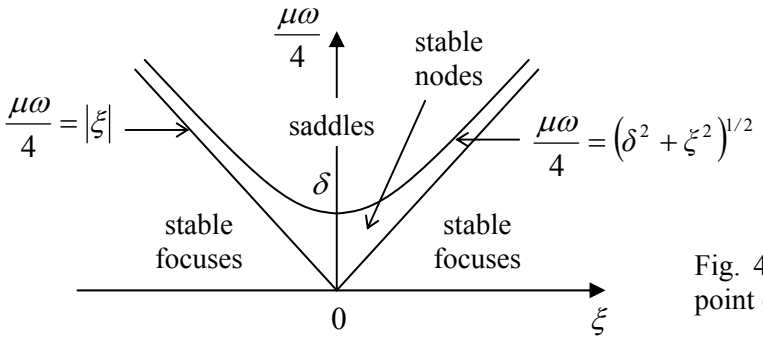


Fig. 4.10. Types of the trivial fixed point of a parametric oscillator.

This double degeneracy of the parametric oscillation's phase could already be noticed from Eqs. (77), because they are evidently invariant with respect to replacement $\varphi \to \varphi + \pi$. Moreover, the degeneracy is not an artifact of the rotating-wave approximation, because the initial Eq. (75) is already invariant with respect to the corresponding replacement $q(t) \to q(t - \pi/\omega)$. This invariance means that all other characteristics (e.g., the amplitude) of the parametric oscillations excited with either of two phases are *absolutely* similar. At the dawn of the computer age (in the late 1950s and early 1960s), there were substantial attempts, especially in Japan, to use this property for storage and processing digital information coded in the phase-binary form.

4.7. Numerical approach

If the amplitude of oscillations, by whatever reason, becomes so large that the nonlinear terms in the equation describing a system are comparable to its linear terms, numerical methods are virtually the only avenue available for their study. In Hamiltonian 1D systems, such methods may be applied directly to integral (3.26), but dissipative and/or parametric systems typically lack first integrals of motion similar to Eq. (3.24), so that the initial differential equation has to be solved.

Let us discuss the general idea of such methods on the example of what mathematicians call the *Cauchy problem* (finding the solution for all moments of time, starting from known initial conditions) for first-order differential equation

$$\dot{q} = f(t, q). \tag{4.96}$$

(The generalization to a set of several such equations is straightforward.) Breaking the time axis into small, equal steps h (Fig. 11) we can reduce the equation integration problem to finding the function value in the next time point, $q_{n+1} \equiv q(t_{n+1}) = q(t_n + h)$ from the previously found value $q_n = q(t_n)$ - and, if

necessary, the values of q at other previous time steps. In the generic approach (called the *Euler method*), q_{n+1} is found using the following formula:

$$\begin{aligned} q_{n+1} &= q_n + k, \\ k &\equiv h f(t_n, q_n). \end{aligned} \tag{4.97}$$

It is evident that this approximation is equivalent to the replacement of the genuine function $q(t)$, on the segment $[t_n, t_{n+1}]$, with the two first terms of its Taylor expansion in point t_n:

$$q(t_n + h) \approx q(t_n) + \dot{q}(t_n)h \equiv q(t_n) + hf(t_n, q_n). \tag{4.98}$$

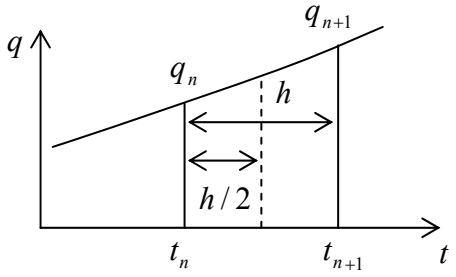


Fig. 4.11. The basic notions used at numerical integration of ordinary differential equations.

Such approximation has an error proportional to h^2. One could argue that making the step h sufficiently small the Euler's method error might be done arbitrary small, but even with the number-crunching power of modern computers, the computation time necessary to reach sufficient accuracy may be too high for large problems.[31] Besides that, the increase of the number of time steps, which is necessary at $h \to 0$, increases the total rounding errors, and eventually may cause an increase, rather than the reduction of the overall error of the computed result.

A more efficient way is to modify Eq. (97) to include the terms of the second order in h. There are several ways to do this, for example using the *2nd-order Runge-Kutta* method:

$$\begin{aligned} q_{n+1} &= q_n + k_2, \\ k_2 &\equiv h f\left(t_n + \frac{h}{2}, q_n + \frac{k_1}{2}\right), \quad k_1 \equiv h f(t_n, q_n). \end{aligned} \tag{4.99}$$

One can readily check that this method gives the exact result if function $q(t)$ is a quadratic polynomial, and hence in the general case its errors are of the order of h^3. We see that the main idea here is to first break the segment $[t_n, t_{n+1}]$ in half (Fig. 11), then evaluate the right-hand part of the differential equation (96) at the point intermediate (in both t and q) between points n and $(n + 1)$, and then use this information to predict q_{n+1}.

The advantage of the Runge-Kutta approach is that it can be readily extended to the 4th order, without an additional breaking of the interval $[t_n, t_{n+1}]$.:

[31] In addition, the Euler method is not time-reversible - the handicap which may be essential for integration of Hamiltonian systems described by systems of second-order differential equations. However, this drawback may be readily overcome by the so-called *leapfrogging* – the overlap of time steps h for a generalized coordinate and the corresponding generalized velocity.

$$q_{n+1} = q_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$
$$k_4 \equiv h f(t_n + h, q_n + k_3),\ k_3 \equiv h f(t_n + \frac{h}{2}, q_n + \frac{k_2}{2}),\ k_2 \equiv h f(t_n + \frac{h}{2}, q_n + \frac{k_1}{2}),\ k_1 \equiv h f(t_n, q_n). \quad (4.100)$$

This method reaches much lower error, $O(h^5)$, without being not too cumbersome. These features have made the 4th-order Runge-Kutta the default method in most numerical libraries. Its extension to higher orders is possible but requires more complex formulas and is justified only for some special cases, e.g., very abrupt functions $q(t)$.[32] The most frequent enhancement of the method is the automatic adjustment of step h to reach the specified accuracy.

Figure 12 shows a typical example of application of that method to the very simple problem of a damped linear oscillator, for two values of fixed time step h (expressed in terms of the number N of such steps per oscillation period). Black lines connect the points obtained by the 4th-order Runge-Kutta method, while the points connected by green straight lines present the exact analytical solution (22). A few-percent errors start to appear only at as few as ~10 time steps per period, so that the method is indeed very efficient. I will illustrate the convenience and handicaps of the numerical approach to the solution of dynamics problems on the discussion of the following topic.

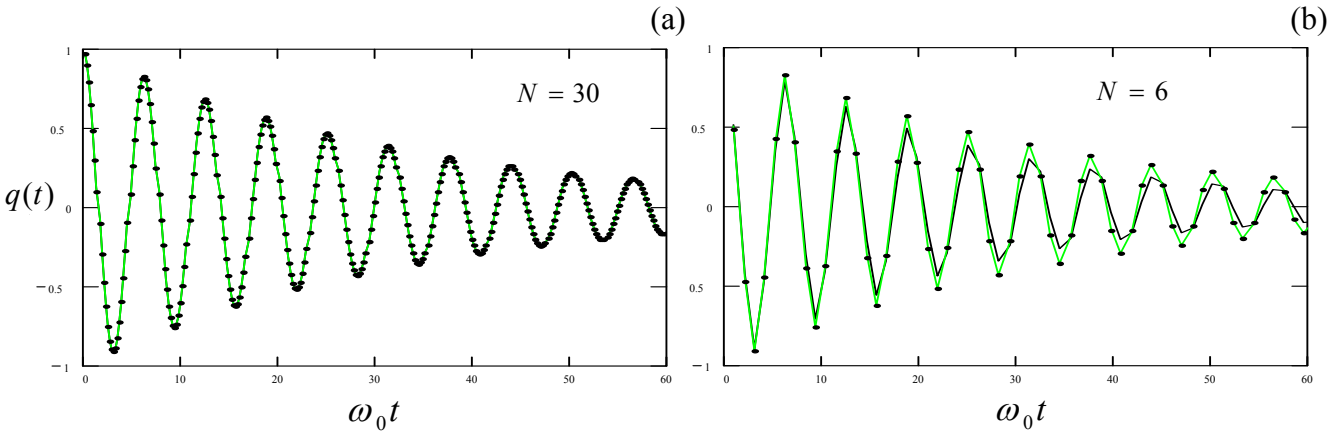


Fig. 4.12. Results of the fixed-point Runge-Kutta solution to the equation of linear oscillator with damping (with $\delta/\omega_0 = 0.03$) for: (a) 30 and (b) 6 points per oscillation period. The results are shown by points; lines are only the guide for the eye.

4.8. Higher harmonic and subharmonic oscillations

Figure 13 shows the numerically calculated[33] transient process and stationary oscillations in a linear oscillator and a very representative nonlinear system, the pendulum described by Eq. (42), both with the same resonance frequency ω_0 for small oscillations. Both systems are driven by a sinusoidal

[32] The most popular approaches in such cases are the *Richardson extrapolation*, the *Bulirsch-Stoer algorithm*, and a set of *prediction-correction techniques*, e.g. the *Adams-Bashforth-Moulton method* – see the literature recommended in MA Sec. 16 (iii).

[33] All numerical results shown in this section have been obtained by the 4th-order Runge-Kutta method with the automatic step adjustment which guarantees the relative error of the order of 10^{-4} – much smaller than the pixel size in the plots.

external force of the same amplitude and frequency - in this illustration, equal to the small-oscillation eigenfrequency ω_0 of both systems. The plots show that despite a very substantial amplitude of the pendulum oscillations (an angle amplitude of about one radian) their waveform remains almost exactly sinusoidal.[34] On the other hand, the nonlinearity affects the oscillation amplitude very substantially. These results illustrate that the validity of the small-parameter method and its RWA extension far exceeds what might be expected from the formal requirement $|q| << 1$.

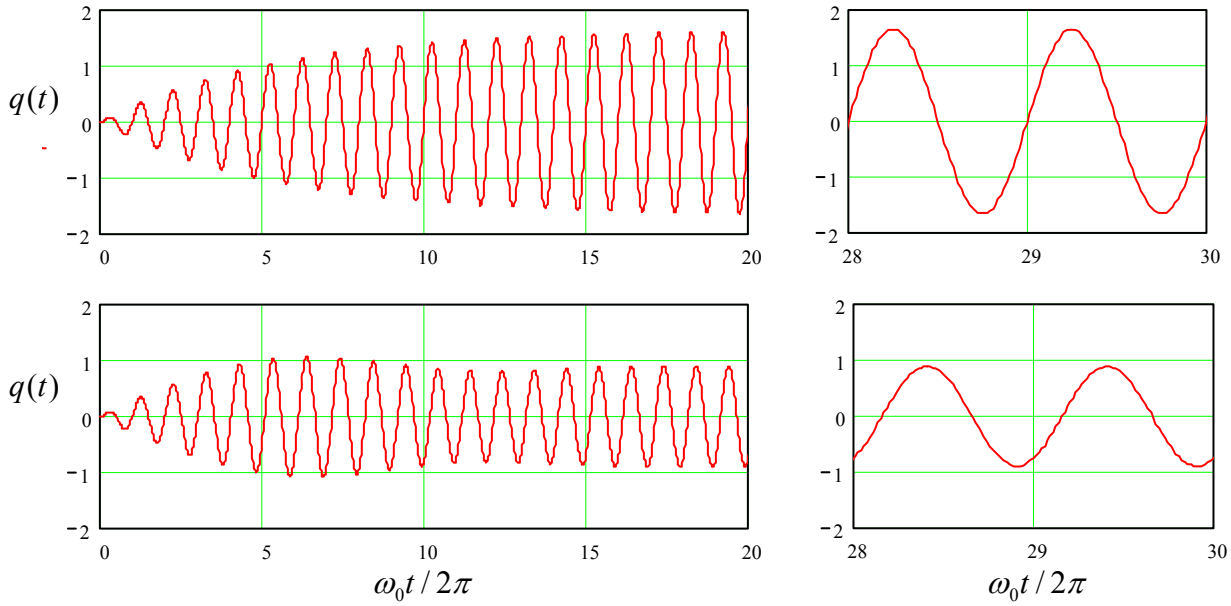


Fig. 4.13. Oscillations induced by a similar sinusoidal external force (turned on at $t = 0$) in two systems with the same small-oscillation frequency ω_0 and low damping – a linear oscillator (two top panels) and a pendulum (two bottom panels). $\delta/\omega_0 = 0.03$, $f_0 = 0.1$, and $\omega = \omega_0$.

The higher harmonic contents in the oscillation waveform may be sharply increased[35] by reducing the external force frequency to $\sim\omega_0/n$, where integer n is the number of the desirable harmonic. For example, Fig. 14a shows oscillations in a pendulum described by the same Eq. (42), but driven at frequency $\omega_0/3$. One can see that the 3rd harmonic amplitude may be comparable with that of the basic harmonic, especially if the external frequency is additionally lowered (Fig. 14b) to accommodate for the deviation of the effective frequency $\omega_0(a)$ of own oscillations from its small-oscillation value ω_0 – see Eq. (49), Fig. 4 and their discussion in Sec. 2 above.

Generally, the higher harmonic generation by nonlinear systems might be readily anticipated. Indeed, the Fourier theorem tells us that any non-sinusoidal periodic function of time, e.g., an initially sinusoidal waveform of frequency ω, distorted by nonlinearity, may be presented as a sum of its basic harmonic and higher harmonics with frequencies $n\omega$. Note that an effective generation of higher

[34] In this particular case, the higher harmonic contents is about 0.5%, dominated by the 3rd harmonic whose amplitude and phase are in a very good agreement with Eq. (50).

[35] This method is used in practice, for example, for the generation of electromagnetic waves with frequencies in the terahertz range (10^{12}-10^{13} Hz) which still lacks efficient electronic self-oscillators.

harmonics is only possible with adequate nonlinearity of the system. For example, consider the nonlinear term αq^3 used in equations explored in Secs. 2 and 3. If the waveform $q(t)$ is approximately sinusoidal, such term can create only the basic and 3rd harmonics. The "pendulum nonlinearity" $\sin q$ cannot produce, without a constant component is process $q(t)$, any even (e.g., the 2nd) harmonic. The most efficient generation of harmonics may be achieved using systems with the sharpest nonlinearities – e.g., semiconductor diodes whose current may follow an exponential dependence on the applied voltage through several orders of magnitude.

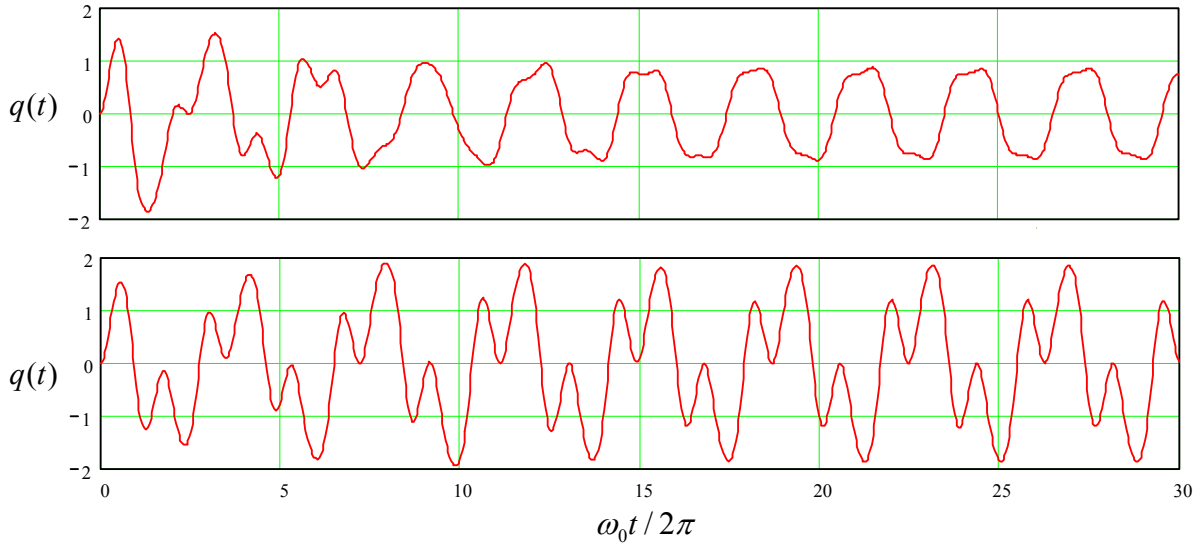


Fig. 4.14. Oscillations induced in a pendulum with damping $\delta/\omega_0 = 0.03$, driven by a sinusoidal external force of amplitude $f_0 = 0.75$, and frequency $\omega_0/3$ (top panel) and $0.8\omega_0/3$ (bottom panel).

However, numerical modeling of nonlinear oscillators, as well as experiments with their physical implementations, bring more surprises. For example, the bottom panel of Fig. 15 shows oscillations in a pendulum under effect of a strong sinusoidal force with a frequency close to $3\omega_0$. One can see that at some parameter values and initial conditions the system's oscillation spectrum is heavily contributed (almost dominated) by the 3rd *sub*harmonic, i.e. a component that is synchronous with the driving force of frequency 3ω, but has the frequency ω that is close to the eigenfrequency ω_0 of the system.

This counter-intuitive phenomenon may be explained as follows. Let us assume that the subharmonic oscillations of frequency $\omega \approx \omega_0$ have somehow appeared, and coexist with the forced oscillations of frequency 3ω:

$$q(t) \approx A\cos\Psi + A_{\text{sub}}\cos\Psi_{\text{sub}}, \quad \text{where } \Psi \equiv 3\omega t - \varphi, \quad \Psi_{\text{sub}} \equiv \omega t - \varphi_{\text{sub}}. \tag{4.101}$$

Then, the first nonlinear term αq^3 of the Taylor expansion of pendulum's nonlinearity $\sin q$ yields

$$\begin{aligned} q^3 &= (A\cos\Psi + A_{\text{sub}}\cos\Psi_{\text{sub}})^3 \\ &= A^3\cos^3\Psi + 3A^2A_{\text{sub}}\cos^2\Psi\cos\Psi_{\text{sub}} + 3AA_{\text{sub}}^2\cos\Psi\cos^2\Psi_{\text{sub}} + A_{\text{sub}}^3\cos^3\Psi_{\text{sub}}. \end{aligned} \tag{4.102}$$

While the first and the last terms of this expression depend only of amplitudes of the individual components of oscillations, the two middle terms are more interesting because they produce so-called *combinational frequencies* of the two components. For our case, the third term,

$$3A A_{\rm sub}^2 \cos\Psi \cos^2 \Psi_{\rm sub} = \frac{3}{4} A A_{\rm sub}^2 \cos(\Psi - 2\Psi_{\rm sub}) + \dots , \tag{4.103}$$

of a special importance, because it produces, besides other combinational frequencies, the subharmonic component with the total phase

$$\Psi - 2\Psi_{\rm sub} = \omega t - \varphi + 2\varphi_{\rm sub} . \tag{4.104}$$

Thus, within a certain range of the mutual phase shift between the Fourier components, this nonlinear contribution is synchronous with the subharmonic oscillations, and describes the interaction that can deliver to it the energy from the external force, so that the oscillations may be self-sustained. Note, however, that the amplitude of the term (103) describing this energy exchange is proportional to the square of $A_{\rm sub}$, and vanishes at the linearization of the equations of motion near the trivial fixed point. This means that the point is always stable, i.e., the 3rd subharmonic cannot be self-excited and always need an initial "kick-off" – compare the two panels of Fig. 15. The same is evidently true for higher subharmonics.

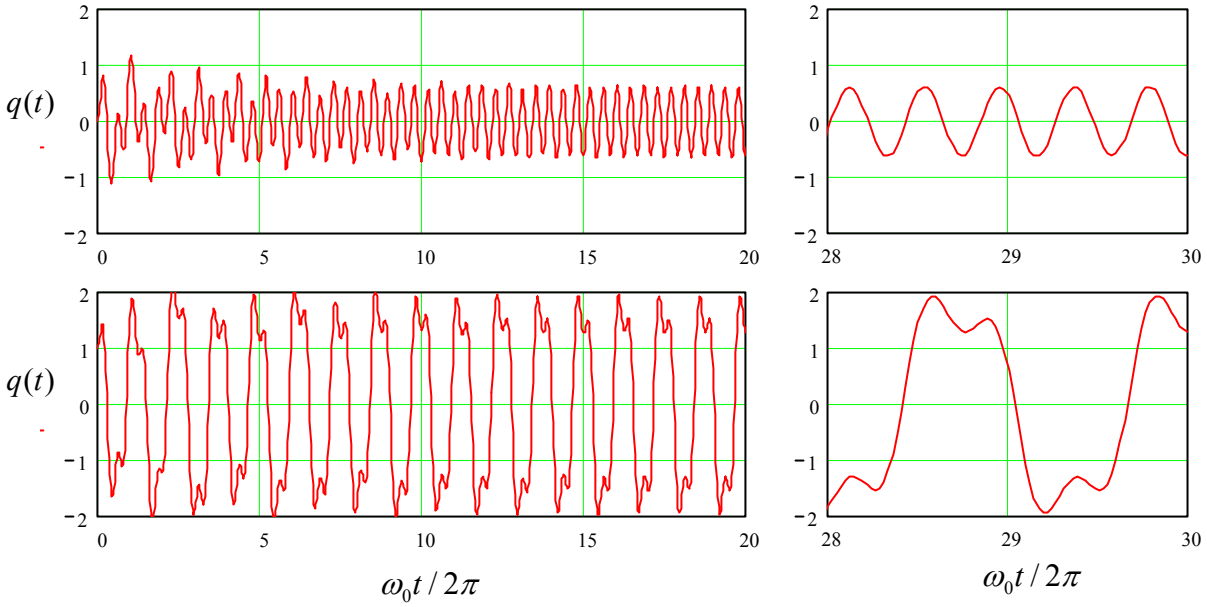


Fig. 4.15. Oscillations induced in a pendulum with $\delta/\omega_0 = 0.03$ by a sinusoidal external force of amplitude $f_0 = 3$ and frequency $3\omega_0 \times 0.8$, with initial conditions $q(0) = 0$ (the top row) and $q(0) = 1$ (the bottom row).

Only the second subharmonic presents a special case. Indeed, let us make a calculation similar to Eq. (102), by replacing Eq. (101) with

$$q(t) \approx A\cos\Psi + A_{\rm sub}\cos\Psi_{\rm sub}, \quad \text{where } \Psi \equiv 2\omega t - \varphi, \quad \Psi_{\rm sub} \equiv \omega t - \varphi_{\rm sub}, \tag{4.105}$$

for a nonlinear term proportional to q^2:

$$q^2 = (A\cos\Psi + A_{\rm sub}\cos\Psi_{\rm sub})^2 = A^2\cos^2\Psi + 2AA_{\rm sub}\cos\Psi\cos\Psi_{\rm sub} + A_{\rm sub}^2\cos^2\Psi_{\rm sub}. \quad (4.106)$$

Here the combinational-frequency term capable of supporting the 2nd subharmonic,

$$2AA_{\rm sub}\cos\Psi\cos\Psi_{\rm sub} = AA_{\rm sub}\cos(\Psi - \Psi_{\rm sub}) = AA_{\rm sub}\cos(\omega t - \varphi + \varphi_{\rm sub}) + \ldots, \quad (4.107)$$

is linear in the subharmonic amplitude, i.e. survives the equation linearization near the trivial fixed point. This mean that the second subharmonic may arise spontaneously, from infinitesimal fluctuations.

Moreover, such excitation of the second subharmonic is very similar to the parametric excitation that was discussed in detail in Sec. 5, and this similarity is not coincidental. Indeed, let us redo expansion (4.106) at a somewhat different assumption that the oscillations are a sum of the forced oscillations at the external force frequency 2ω, and an *arbitrary but weak* perturbation:

$$q(t) = A\cos(2\omega t - \varphi) + \tilde{q}(t), \quad |\tilde{q}| << A. \quad (4.108)$$

Then, neglecting the small term proportional to $\tilde{q}^2$, we get

$$q^2 \approx A^2\cos^2(2\omega t - \varphi) + 2\tilde{q}(t)A\cos(2\omega t - \varphi). \quad (4.109)$$

Besides the inconsequential phase φ, the second term in the last formula is *exactly* similar to the term describing the parametric effects in Eq. (75). This fact means that for a weak perturbation, a system with a quadratic nonlinearity in the presence of a strong "pumping" signal of frequency 2ω is equivalent to a system with parameters changing in time with frequency 2ω. This fact is broadly used for the parametric excitation at high (e.g., optical) frequencies where the mechanical means of parameter modulation (see, e.g., Fig. 5) are not practicable. The necessary quadratic nonlinearity at optical frequencies may be provided by a *noncentrosymmetric nonlinear crystal*, e.g., the β-phase barium borate (BaB_2O_4).

Before finishing this chapter, let me elaborate a bit on a general topic: the relation between the numerical and analytical approaches to problems of dynamics (and physics as a whole). We have just seen that sometimes numerical solutions, like those shown in Fig. 15b, may give vital clues for previously unanticipated phenomena such as the excitation of subharmonics. (The phenomenon of deterministic chaos, which will be discussed in Chapter 9 below, presents another example of such "numerical discoveries".) One might also argue that in the absence of exact analytical solutions, numerical simulations may be the main theoretical tool for the study of such phenomena. These hopes are, however, muted by the problem that is frequently called the *curse of dimensionality*,[36] in which the last word refers to the number of input parameters of the problem to be solved.[37]

Indeed, let us have another look at Fig. 15. OK, we have been lucky to find a new phenomenon, the 3rd subharmonic generation, for a particular set of parameters - in that case, five of them: $\delta/\omega_0 = 0.03$, $3\omega/\omega_0 = 2.4$, $f_0 = 3$, $q(0) = 1$, and $dq/dt\ (0) = 0$. Could we tell anything about how common this effect is? Are subharmonics with different n possible in the system? The only way to address these

[36] This term had been coined in 1957 by R. Bellman in the context of the optimal control theory (where the dimensionality typically means the number of parameters affecting the system under control), but gradually has spread all over quantitative sciences using numerical methods.

[37] In EM Sec. 1.2, I discuss implications of the curse implications for a different case, when both analytical and numerical solutions to the same problem are possible.

questions computationally is to carry out similar numerical simulations in many points of the d-dimensional (in this case, $d = 5$) space of parameters. Say, we have decided that breaking the reasonable range of each parameter to $N = 100$ points is sufficient. (For many problems, even more points are necessary – see, e.g., Sec. 9.1.) Then the total number of numerical experiments to carry out is $N^d = (10^2)^5 = 10^{10}$ – not a simple task even for the powerful modern computing facilities. (Besides the pure number of required CPU cycles, consider storage and analysis of the results.) For many important problems of nonlinear dynamics, e.g., turbulence, the parameter dimensionality d is substantially larger, and the computer resources necessary for one numerical experiment, are much greater.

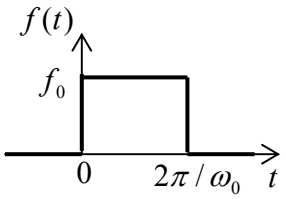
In the view of the curse of dimensionality concerns, approximate analytical considerations, like those outlined above for the subharmonic excitation, are invaluable. More generally, physics used to stand on two legs, experiment and (analytical) theory. The enormous progress of computer performance during a few last decades has provided it with one more point of support (a tail? :-) – numerical simulation. This does not mean we can afford to cut and throw away any of the legs we are standing on.

4.9. Exercise problems

4.1.* Prove Eq. (26) for the response function given by Eq. (17).

Hint: You may like to use the *Cauchy integral theorem* for analytical functions of complex variable.[38]

4.2. A square-wave pulse of force (see Fig. on the right) is exerted on a linear oscillator with eigenfrequency ω_0 (no damping), initially at rest. Calculate the law of motion $q(t)$, sketch it, and interpret the result.



4.3. At $t = 0$, a sinusoidal external force $F(t) = F_0 \cos \omega t$, with constant A and ω, is applied to a linear oscillator with eigenfrequency ω_0 and damping δ, which was at rest at $t \le 0$.

(i) Calculate the general expression for the time evolution of the oscillator's coordinate, and interpret the result.

(ii) Spell out your result for the case of the resonance ($\omega = \omega_0$) in a system with low damping ($\delta << \omega_0$), and, in particular, explore the limit $\delta \to 0$.

4.4. A pulse of external force $F(t)$, with a finite duration $\mathcal{T}$, is exerted on a harmonic oscillator, initially at rest in the equilibrium position. Neglecting dissipation, calculate the change of oscillator's energy, using two different methods, and compare the results.

4.5.* For a system with the following Lagrangian function:

$$L = \frac{m}{2}\dot{q}^2 - \frac{\kappa}{2}q^2 + \frac{\varepsilon}{2}\dot{q}^2 q^2,$$

calculate the frequency of free oscillations as a function of their amplitude A, at $A \to 0$, using two different approaches.

[38] See, e.g., MA Eq. (15.1).

4.6. For a system with the Lagrangian function

$$L = \frac{m}{2}\dot{q}^2 - \frac{\kappa}{2}q^2 + \varepsilon\dot{q}^4,$$

with small parameter ε, use the rotating-wave approximation to find the frequency of free oscillations as a function of their amplitude.

4.7. Find the regions of real, time-independent parameters a_1 and a_2, in which the fixed point of the following system of equations,

$$\dot{q}_1 = a_1(q_2 - q_1),$$
$$\dot{q}_2 = a_2 q_1 - q_2,$$

is unstable. On the $[a_1, a_2]$ plane, sketch the regions of each fixed point type - stable and unstable nodes, focuses, etc.

4.8. Solve Problem 4.3(ii) using the rotating-wave approximation, and compare the result with the exact solution.

4.9. Use the rotating-wave approximation to analyze forced oscillations in an oscillator with weak nonlinear damping, described by equation

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q + \beta\dot{q}^3 = f_0\cos\omega t,$$

with $\omega \approx \omega_0$; β, $\delta > 0$; and $\beta\omega A^2 << 1$. In particular, find the stationary amplitude of forced oscillations and analyze their stability. Discuss the effect(s) of the nonlinear term on the resonance.

4.10.* Analyze stability of the forced nonlinear oscillations described by Eq. (43). Relate the result to the slope of resonance curves (Fig. 4).

4.11. Use the rotating-wave approximation to analyze parametric excitation of an oscillator with weak nonlinear damping, described by equation

$$\ddot{q} + 2\delta\dot{q} + \beta\dot{q}^3 + \omega_0^2(1 + \mu\cos 2\omega t)q = 0,$$

with $\omega \approx \omega_0$; β, $\delta > 0$; and μ, $\beta\omega A^2 << 1$. In particular, find the amplitude of stationary oscillations and analyze their stability.

4.12. Adding nonlinear term αq^3 to the left-hand part of Eq. (76),

(i) find the corresponding addition to the RWA equations,
(ii) find the stationary amplitude A of parametric oscillations,
(iii) sketch and discuss the $A(\xi)$ dependence,
(iv) find the type and stability of each fixed point of the RWA equations,
(v) sketch the Poincaré phase plane of the system in main parameter regions.

4.13. Use the rotating-wave approximation to find the conditions of parametric excitation in an oscillator with weak modulation of both the effective mass $m(t) = m_0(1 + \mu_m\cos 2\omega t)$ and spring constant $\kappa(t) = \kappa_0[1 + \mu_\kappa\cos(2\omega t - \psi)]$, with the same frequency $2\omega \approx 2\omega_0$, but arbitrary modulation depths ratio

μ_m/μ_k and phase shift ψ. Interpret the result in terms of modulation of the instantaneous frequency $\omega(t) \equiv [\kappa(t)/m(t)]^{1/2}$ and mechanical impedance $Z(t) \equiv [\kappa(t)m(t)]^{1/2}$ of the oscillator.

4.14.* Find the condition of parametric excitation of a nonlinear oscillator described by equation

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q + \gamma q^2 = f_0 \cos 2\omega t,$$

with sufficiently small δ, γ, f_0, and $\xi \equiv \omega - \omega_0$.