Chapter 1. Review of Fundamentals

After elaborating a bit on the title and contents of the course, this short introductory chapter lists the basic notions and facts of the classical mechanics, that are supposed to be known to the reader from undergraduate studies.[1] *Due to this reason, the explanations are very brief.*

1.1. Mechanics and dynamics

A more fair title of this course would be *Classical Mechanics and Dynamics*, because the notions of mechanics and dynamics, though much intertwined, are still somewhat different. Term m*echanics*, in its narrow sense, means deriving the equations of motion of point-like particles and their systems (including solids and fluids), solution of these equations, and interpretation of the results. *Dynamics* is a more ambiguous term; it may mean, in particular:

(i) the part of mechanics that deals with motion (in contrast to *statics*);
(ii) the part of mechanics that deals with reasons for motion (in contrast to *kinematics*);
(iii) the part of mechanics that focuses on its two last tasks, i.e. the solution of the equations of motion and discussion of the results.

The last definition invites a question. It may look that mechanics and dynamics are just two sequential steps of a single process; why should they be considered separate disciplines? The main reason is that the many differential equations of motion, obtained in classical mechanics, also describe processes in different systems, so that their analysis may reveal important features of these systems as well. For example, the famous ordinary differential equation

$$\ddot{x} + \omega_0^2 x = 0 \tag{1.1}$$

describes sinusoidal 1D oscillations not only of a mass on a spring, but also of an electric or magnetic field in a resonator, and many other systems. Similarly, the well-known partial differential equation

$$\left(\frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) f(\mathbf{r}, t) = 0\,, \tag{1.2}$$

where v is a constant and ∇^2 is the Laplace operator,[2] describes not only acoustic waves in an elastic mechanical continuum (solid or fluid), but also electromagnetic waves in a non-dispersive media, certain chemical reactions, etc. Thus the results of analysis of the dynamics described by these equations may be reused for applications well beyond mechanics.

[1] The reader is advised to perform a self-check by solving a few problems of the dozen listed in Sec. 1.7. If the results are not satisfactory, it may make sense to start from some remedial reading. For that, I could recommend, for example (in the alphabetical order): G. R. Fowles and G. L. Cassiday, *Analytical Mechanics*, 7th ed., Brooks Cole, 2004; K. R. Symon, *Mechanics*, 3rd ed., Addison-Wesley, 1971; or J. B. Marion and S. T. Thornton, *Classical Dynamics of Particles and Systems,* 4th ed., Saunders, 1995.

[2] This series assumes reader's familiarity with the basic calculus and vector algebra. The formulas most important for this series are listed in the *Selected Mathematical Formulas* appendix, referred below as MA. In particular, a reminder of the definition and the basic properties of the Laplace operator may be found in MA Sec. 9.

To summarize, term "dynamics" is so ambiguous[3] that, after some hesitation, I have opted to using for this course the traditional name *Classical Mechanics*, implying its broader meaning, which includes (similarly to *Quantum Mechanics* and *Statistical Mechanics*) studies of dynamics of some non-mechanical systems.

1.2. Kinematics: Basic notions

The basic notions of kinematics may be defined in various ways, and some mathematicians pay a lot of attention to analyzing such systems of axioms and relations between them. In physics, we typically stick to less rigorous ways (in order to proceed faster to particular problems), and end debating a definition as soon as everybody in the room agrees that we are all speaking about the same thing. Let me hope that the following notions used in classical mechanics do satisfy this criterion:

(i) All the *Euclidean geometry* notions, including the *geometric point* (the mathematical abstraction for the position of a very small object), straight line, etc.

(ii) The *orthogonal, linear* ("Cartesian") *coordinates*[4] r_j of a geometric point in a particular *reference frame* – see Fig. 1.[5]

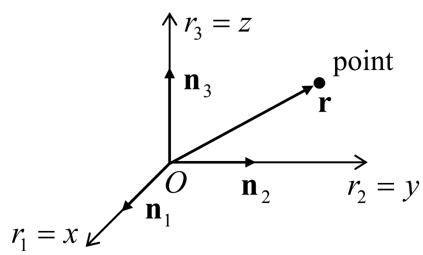

Fig. 1.1. Cartesian coordinates and radius-vector of a point/particle.

The coordinates may be used to define the point's *radius-vector*[6]

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[3] Another important issue is: Definition (iii) of dynamics is suspiciously close to the part of mathematics devoted to the differential equation analysis; what is the difference? To answer, we have to dip, for just a second, into the philosophy of physics. Physics may be described as an art (and a bit of science :-) of description of Mother Nature by mathematical means; hence in many cases the approaches of a mathematician and a physicist to a problem are very similar. The main difference is that physicists try to express the results of their analysis in terms of *system's motion* rather than *function properties*, and as a result develop some sort of intuition ("gut feeling") about how other, apparently similar, systems may behave, even if their exact equations of motion are somewhat different - or not known at all. The intuition so developed has an enormous heuristic power, and most discoveries in physics have been made through gut-feeling-based insights rather than by plugging one formula into another one.

[4] In these notes the Cartesian coordinates are denoted either as either $\{r_1, r_2, r_3\}$ or $\{x, y, z\}$, depending on convenience in the particular case. Note that axis numbering is important for operations like the vector ("cross") product; the "correct" (meaning generally accepted) numbering order is such that rotation $\mathbf{n}_1 \to \mathbf{n}_2 \to \mathbf{n}_3 \to \mathbf{n}_1\ldots$ looks counterclockwise if watched from a point with all $r_i > 0$ – see Fig. 1.

[5] In references to figures, formulas, problems and sections within the same chapter of these notes, the chapter number is dropped for brevity.

[6] From the point of view of the tensor theory (in which the physical vectors like $\mathbf{r}$ are considered the *rank-1 tensors*), it would be more natural to use superscripts in the components r_j and other "contravariant" vectors. However, the superscripts may be readily confused with the power signs, and I will postpone this notation (as well as the implied summation over the repeated indices) until the discussion of relativity in EM Chapter 9.

Radius -vector

$$\mathbf{r} = \sum_{j=1}^{3} \mathbf{n}_j r_j \,, \tag{1.3}$$

where $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are the unit vectors along coordinate axis directions, with the *Euclidean metric*:

Euclidean metric

$$r^2 = \sum_{j=1}^{3} r_j^2 \,. \tag{1.4}$$

which is independent, in particular, of the distribution of matter in space.

(iii) The *time* – as described by a continuous scalar variable (say, t), typically considered an independent argument of various physical observables, in particular the point's radius-vector $\mathbf{r}(t)$. By accepting Eq. (4), and an implicit assumption that time t runs similarly in all reference frames, we subscribe to the notion of the *absolute* ("Newtonian") *space/time*, and hence abstain from a discussion of relativistic effects.[7]

(iv) The (instant) *velocity* of the point,

Velocity

$$\mathbf{v}(t) \equiv \frac{d\mathbf{r}}{dt} \equiv \dot{\mathbf{r}} \,, \tag{1.5}$$

and its *acceleration*:

Acceleration

$$\mathbf{a}(t) \equiv \frac{d\mathbf{v}}{dt} \equiv \dot{\mathbf{v}} = \ddot{\mathbf{r}} \,. \tag{1.6}$$

Since the above definitions of vectors **r**, **v**, and **a** depend on the chosen reference frame (are "reference-frame-specific"), there is a need to relate those vectors as observed in different frames. Within the Euclidean geometry, for two reference frames with the corresponding axes parallel in the moment of interest (Fig. 2), the relation between the radius-vectors is very simple:

Radius-vector's transformation

$$\mathbf{r}\big|_{\text{in } O'} = \mathbf{r}\big|_{\text{in } O} + \mathbf{r}_O\big|_{\text{in } O'} \,. \tag{1.7}$$

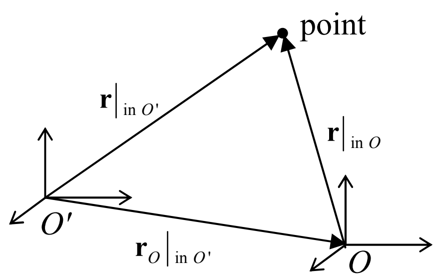

Fig. 1.2. Coordinate transfer between two reference frames.

[7] Following tradition, an introduction to special relativity is included into the *Classical Electrodynamics* ("EM") part of these notes. The relativistic effects are small if all particles velocities are much lower than the speed of light, $c \approx 3.00\times10^8$ m/s, and all distances are much larger then the system's *Schwarzschild radius* $r_s \equiv 2Gm/c^2$, where $G \approx 6.67\times10^{-11}$ SI units (m^3/kg·s) is the Newtonian gravity constant, and m is system's mass. (More exact values of c, G, and some other physical constants may be found in appendix *CA: Selected Physical Constants*.)

If the frames move versus each other by *translation* only (no mutual rotation!), similar relations are valid for velocity and acceleration as well:

$$\mathbf{v}\big|_{\text{in } O'} = \mathbf{v}\big|_{\text{in } O} + \mathbf{v}_O\big|_{\text{in } O'}, \tag{1.8}$$

$$\mathbf{a}\big|_{\text{in } O'} = \mathbf{a}\big|_{\text{in } O} + \mathbf{a}_O\big|_{\text{in } O'}. \tag{1.9}$$

In the case of mutual rotation of the reference frames, notions like $\mathbf{v}_O\big|_{\text{in } O'}$ are not well defined. (Indeed, different points of a rigid body connected to frame O may have different velocities in frame O'.) As a result, the transfer laws for velocities and accelerations are more complex than those given by Eqs. (8) and (9). It will be more natural for me to discuss them in the end of Chapter 5 that is devoted to rigid body motion.

(v) The *particle*: a localized physical object whose size is negligible, and shape unimportant *for the given problem*. Note that the last qualification is extremely important. For example, the size and shape of a Space Shuttle are not too important for the discussion of its orbital motion, but are paramount when its landing procedures are being developed. Since classical mechanics neglects the quantum mechanical uncertainties,[8] particle's position, at any particular instant t, may be identified with a single geometric point, i.e. one radius-vector $\mathbf{r}(t)$. Finding the *laws of motion* $\mathbf{r}(t)$ of all particles participating in the given problem is frequently considered the final goal of classical mechanics.

1.3. Dynamics: Newton laws

Generally, the classical dynamics is fully described (in addition to the kinematic relations given above) by three *Newton laws*.[9] In contrast to the impression some textbooks on theoretical physics try to create, these laws are experimental in nature, and cannot be derived from *purely* theoretical arguments.[10]

I am confident that the reader of these notes is already familiar with the Newton laws, in one or another formulation. Let me note only that in some formulations the *1st Newton law* looks just as a particular case of the 2nd law - for the case of zero net force acting on a particle. In order to avoid this duplication, the 1st law may be formulated as the following postulate:

- There exists at least one reference frame, called *inertial*, in which any *free particle* (i.e. a particle isolated from the rest of the Universe) moves with $\mathbf{v}$ = const, i.e. with $\mathbf{a} = 0$. 1st Newton law

According to Eq. (9), this postulate immediately means that there is also an infinite number of inertial frames, because all frames O' moving without rotation or acceleration relative to the postulated inertial frame O (i.e. having $\mathbf{a}_O\big|_{\text{in } O'} = 0$) are also inertial.

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8 This approximation is legitimate, crudely, when the product of the coordinate and momentum scales of the particle motion is much larger than the Planck's constant $\hbar \approx 1.054\times10^{-34}$ J·s. A more exact formulation may be found, e.g., in the *Quantum Mechanics* ("QM") part of these note series.

9 Due to the genius of Sir Isaac Newton, these laws were formulated as early as in 1687, far ahead of the science of that time.

10 Some laws of Nature (including the Newton laws) may be derived from certain more general postulates, such as the *Hamilton* (or "least action") *principle* - see Sec. 10.2 below. Note, however, that such derivations are only acceptable because all known corollaries of the postulates comply with all known experimental results.

On the other hand, the 2nd and 3rd Newton laws may be postulated *together* in the following elegant way. Each particle, say number k, may be characterized by a scalar constant (called *mass* m_k), such that at any interaction of N particles (isolated from the rest of the Universe), in any inertial system,

Total momentum and its conservation

$$\mathbf{P} \equiv \sum_{k=1}^{N} m_k \mathbf{v}_k = \text{const.} \tag{1.10}$$

(Each component of this sum,

Particle's momentum

$$\mathbf{p}_k \equiv m_k \mathbf{v}_k, \tag{1.11}$$

is called the *mechanical momentum* of the corresponding particle, and the whole sum **P**, the *total momentum* of the system.)

Let us apply this postulate to just two interacting particles. Differentiating Eq. (10), written for this case, over time, we get

$$\dot{\mathbf{p}}_1 = -\dot{\mathbf{p}}_2. \tag{1.12}$$

Let us give the derivative $\dot{\mathbf{p}}_1$ (i.e., a vector) the name of *force* $\mathbf{F}_1$ excerted on particle 1. In our current case, when the only possible source of force is particle 2, the force may be denoted as $\mathbf{F}_{12}$. Similarly, $\mathbf{F}_{21} \equiv \dot{\mathbf{p}}_2$, so that we get the *3rd Newton law*

3rd Newton law

$$\mathbf{F}_{12} = -\mathbf{F}_{21}. \tag{1.13}$$

Now, returning to the general case of several interacting particles, we see that an additional (but very natural) assumption that all partial forces $\mathbf{F}_{kk'}$ acting on particle k add up as vectors, leads to the general form of the *2nd Newton law*[11]

2nd Newton law

$$m_k \mathbf{a}_k \equiv \dot{\mathbf{p}}_k = \sum_{k' \neq k} \mathbf{F}_{kk'} \equiv \mathbf{F}_k, \tag{1.14}$$

that allows a clear interpretation of the mass as a measure of particle's *inertia*.

As a matter of principle, if the dependence of all pair forces $\mathbf{F}_{kk'}$ of particle positions (and generally maybe of time as well) is known, Eq. (14) augmented with kinematic relations (4) and (5), allows the calculation of the laws of motion $\mathbf{r}_k(t)$ of all particles of the system. For example, for one particle the 2nd law (14) gives the ordinary differential equation of the second order,

$$m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, t), \tag{1.15}$$

that may be integrated – either analytically or numerically.

For certain cases, this is very simple. As an elementary example, the Newton's gravity field

Newton's gravity law

$$\mathbf{F} = -G\frac{mm'}{R^3}\mathbf{R} \tag{1.16a}$$

(where $\mathbf{R} \equiv \mathbf{r} - \mathbf{r}'$ is the distance between particles of masses m and m')[12], is virtually uniform and may be approximated as

[11] Of course, for composite bodies of varying mass (e.g., rockets emitting jets, see Problem 11), momentum's derivative may differ from $m\mathbf{a}$.

$$\mathbf{F} = m\mathbf{g}, \qquad (1.16b)$$

Uniform gravity field

with the vector $\mathbf{g} \equiv (Gm'/r'^3)\mathbf{r}'$ being constant, for local, relatively small-scale motions, with $r << r'$.[13] As a result, m in Eq. (15) cancels, it is reduced to just $\ddot{\mathbf{r}} = \mathbf{g}$, and may be easily integrated twice:

$$\dot{\mathbf{r}}(t) \equiv \mathbf{v}(t) = \int_0^t \mathbf{g}\, dt' + \mathbf{v}(0) = \mathbf{g}t + \mathbf{v}(0), \qquad (1.17)$$

$$\mathbf{r}(t) = \int_0^t \mathbf{v}(t')dt' + \mathbf{r}(0) = \mathbf{g}\frac{t^2}{2} + \mathbf{v}(0)t + \mathbf{r}(0)\ , \qquad (1.18)$$

thus giving the full solution of all those undergraduate problems on the projectile motion, which should be so familiar to the reader.

All this looks (and indeed is) very simple, but in most other cases leads to more complex calculations. As an example, let us consider another simple problem: a bead of mass m sliding, without friction, along a round ring of radius R in a gravity field obeying Eq. (16b) – see Fig. 3.

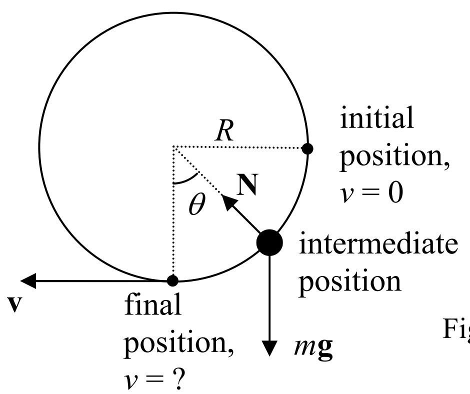

Fig. 1.3. Bead moving on a vertical ring.

Suppose we are only interested in bead's velocity v in the lowest point, after it has been dropped from the rest at the rightmost position. If we want to solve this problem using only the Newton laws, we have to do the following steps:

(i) consider the bead in an arbitrary intermediate position on a ring, described, for example by the angle θ shown in Fig. 3;

(ii) draw all the forces acting on the particle - in our current case, the gravity force $m\mathbf{g}$ and the reaction force $\mathbf{N}$ exerted by the ring;

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[12] Note that the fact that the masses participating in Eqs. (14) and (16) are equal, the so-called *weak equivalence principle*, is highly nontrivial, but has been verified experimentally to the relative accuracy of at least 10^{-13}. Due to its conceptual significance of the principle, new space experiments, such as MISCROSCOPE (http://smsc.cnes.fr/MICROSCOPE/), are being planned for a substantial, nearly 100-fold accuracy improvement.

[13] Of course, the most important particular case of Eq. (1.16b) is the motion of objects near Earth's surface. In this case, using the fact that (1.16a) remains valid for the gravity field created by a heavy sphere, we get $g = GM_E/R_E^2$, where M_E and R_E are the Earth mass and radius. Plugging in their values, $M_E \approx 5.92\times10^{24}$ kg, $R_E \approx 6.37\times10^6$ m, we get $g \approx 9.74$ m/s^2. The effective value of g varies from 9.78 to 9.83 m/s^2 at various locations on Earth's surface (due to the deviations of Earth's shape from a sphere, and the location-dependent effect of the centrifugal "inertial force" – see Sec. 6.5 below), with an average value of $g \approx 9.807$ m/s^2 .

(iii) write the 2nd Newton law for two nonvanishing components of the bead acceleration, say for its vertical and horizontal components a_x and a_y;

(iv) recognize that in the absence of friction, the force **N** should be normal to the ring, so that we can use two additional equations, $N_x = -N\sin\theta$ and $N_y = N\cos\theta$;

(v) eliminate unknown variables N, N_x, and N_y from the resulting system of four equations, thus getting a single second-order differential equation for one variable, for example θ;

(vi) integrate this equation once to get the expression relating the velocity $\dot{\theta}$ and the angle θ; and, finally,

(vii) using our specific initial condition ($\dot{\theta} = 0$ at $\theta = \pi/2$), find the final velocity as $v = R\dot{\theta}$ at $\theta = 0$.

All this is very much doable, but please agree that the procedure it too cumbersome for such a simple problem. Moreover, in many other cases even writing equations of motion along relevant coordinates is very complex, and any help the general theory may provide is highly valuable. In many cases, such help is given by *conservation laws*; let us review the most general of them.

1.4. Conservation laws

(i) *Energy* conservation is arguably the most general law of physics, but in mechanics it takes a more humble form of *mechanical energy conservation* that has limited applicability. To derive it, we first have to define the *kinetic energy* of a particle as

Kinetic energy

$$T \equiv \frac{m}{2}v^2, \tag{1.19}$$

and then recast its differential as[14]

$$dT = d\left(\frac{m}{2}v^2\right) = d\left(\frac{m}{2}\mathbf{v}\cdot\mathbf{v}\right) = m\mathbf{v}\cdot d\mathbf{v} = m\frac{d\mathbf{v}\cdot d\mathbf{r}}{dt} = \frac{d\mathbf{p}}{dt}\cdot d\mathbf{r}. \tag{1.20}$$

Now plugging in the momentum's derivative from the 2nd Newton law, $d\mathbf{p}/dt = \mathbf{F}$, where **F** is the full force acting on the particle, we get relation $dT = \mathbf{F}\cdot d\mathbf{r}$. Its integration along particle's trajectory between some points A and B gives the relation that is sometimes called the *work-energy principle*:

Energy-work principle

$$\Delta T \equiv T(\mathbf{r}_B) - T(\mathbf{r}_A) = \int_A^B \mathbf{F}\cdot d\mathbf{r}, \tag{1.21}$$

where the integral in the right-hand part is called the *work* of the force **F** on the path from A to B.

The further step may be made only for *potential* (also called "conservative") forces that may be presented as (minus) gradients of some scalar function $U(\mathbf{r})$, called the *potential energy*.[15] The vector operator ∇ (called either *del* or *nabla*) of spatial differentiation[16] allows a very compact expression of this fact:

[14] Symbol $\mathbf{a}\cdot\mathbf{b}$ denotes the scalar (or "dot-") product of vectors **a** and **b** - see, e.g., MA Eq. (7.1).

[15] Note that because of its definition via the gradient, the potential energy is only defined to an arbitrary additive constant.

[16] Its basic properties are listed in MA Sec. 8.

$$\boxed{\mathbf{F} = -\nabla U \,.} \tag{1.22}$$

Potential energy

For example, for the uniform gravity field (16b),

$$U = mgh + \text{const}, \tag{1.23}$$

where h is the vertical coordinate directed "up" - opposite to the direction of the vector **g**.

Integrating the tangential component F_τ of the vector **F**, given by Eq. (22), along an arbitrary path connecting points A and B, we get

$$\int_A^B F_\tau dr \equiv \int_A^B \mathbf{F}\cdot d\mathbf{r} = U(\mathbf{r}_A) - U(\mathbf{r}_B)\,, \tag{1.24}$$

i.e. work of potential forces may be presented as the difference of values of function $U(\mathbf{r})$ in the initial and final point of the path. (Note that according to Eq. (24), work of a potential force on any closed trajectory, with $\mathbf{r}_A = \mathbf{r}_B$, is zero.)

Now returning to Eq. (21) and comparing it with Eq. (24), we see that

$$T(\mathbf{r}_B) - T(\mathbf{r}_A) = U(\mathbf{r}_A) - U(\mathbf{r}_B)\,, \tag{1.25}$$

so that the *total mechanical energy* E, defined as

$$\boxed{E \equiv T + U\,,} \tag{1.26}$$

Total mechanical energy

is indeed conserved:

$$\boxed{E(\mathbf{r}_A) \equiv E(\mathbf{r}_B)\,,} \tag{1.27}$$

Mechanical energy conservation

but for conservative forces only. (Non-conservative forces, e.g., friction, typically transfer energy from the mechanical form into some other form, e.g., heat.)

The mechanical energy conservation allows us to return for a second to the problem shown in Fig. 3 and solve it in one shot by writing Eq. (27) for the initial and final points:[17]

$$0 + mgR = \frac{m}{2}v^2 + 0. \tag{1.28}$$

Solving Eq. (28) for v immediately gives us the desired answer. Let me hope that the reader agrees that this way of problem solution is much simpler, and I have got his or her attention to discuss other conservation laws – which may be equally effective.

(ii) <u>*Momentum*</u>. Actually, the conservation of the full momentum of any system of particles isolated from the rest of the world, has already been discussed and may serve as the basic postulate of classical dynamics – see Eq. (10). In the case of one free particle the law is reduced to a trivial result **p** = const, i.e. **v** = const. If the system of N particles *is* affected by external forces $\mathbf{F}^{(\text{ext})}$, we may write

$$\mathbf{F}_k = \mathbf{F}_k^{(\text{ext})} + \sum_{k=1}^{N} \mathbf{F}_{kk'}\,. \tag{1.29}$$

[17] Here the arbitrary constant in Eq. (32) is chosen so that the potential energy is zero in the finite point.

If we sum up the resulting Eqs. (14) for all particles of the system then, due to the 3rd Newton law (13), the contributions of all internal forces to this double sum in the right-hand part cancel, and we get the equation

System's momentum evolution

$$\dot{\mathbf{P}} = \mathbf{F}^{(\text{ext})}, \quad \text{where } \mathbf{F}^{(ext)} \equiv \sum_{k=1}^{N} \mathbf{F}_k^{(\text{ext})}, \tag{1.30}$$

which tells us that the translational motion of the system as the whole is similar to that of a single particle, under the effect of the *net external force* $\mathbf{F}^{(\text{ext})}$. As a simple sanity check, if the external forces have a zero sum, we return to postulate (10). Just one reminder: Eq. (30), just as its precursor Eq. (14), is only valid in an inertial reference frame.

(iii) *Angular momentum* of a particle[18] is defined as the following vector:

Angular momentum: definition

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p}, \tag{1.31}$$

where $\mathbf{a}\times\mathbf{b}$ means the vector (or "cross-") product of the vector operands.[19] Now, differentiating Eq. (31) over time, we get

$$\dot{\mathbf{L}} = \dot{\mathbf{r}} \times \mathbf{p} + \mathbf{r} \times \dot{\mathbf{p}}. \tag{1.32}$$

In the first product, $\dot{\mathbf{r}}$ is just the velocity vector $\mathbf{v}$ which is parallel to the particle momentum $\mathbf{p} = m\mathbf{v}$, so that this product vanishes, since the vector product of any two parallel vectors is zero. In the second product, $\dot{\mathbf{p}}$ equals the full force $\mathbf{F}$ acting on the particle, so that Eq. (32) is reduced to

Angular momentum: evolution

$$\dot{\mathbf{L}} = \boldsymbol{\tau}, \tag{1.33}$$

where vector

Torque

$$\boldsymbol{\tau} \equiv \mathbf{r} \times \mathbf{F}, \tag{1.34}$$

is called the *torque* of force $\mathbf{F}$. (Note that the torque is evidently reference-frame specific - and again, the frame has to be inertial for Eq. (33) to be valid.) For an important particular case of a *central* force $\mathbf{F}$ that is parallel to the radius vector $\mathbf{r}$ of a particle (as measured from the force source point), the torque vanishes, so that (in that particular reference frame only!) the angular momentum is conserved:

Angular momentum: conservation

$$\mathbf{L} = \text{const.} \tag{1.35}$$

For a system of N particles, the total angular momentum is naturally defined as

$$\mathbf{L} \equiv \sum_{k=1}^{N} \mathbf{L}_k. \tag{1.36}$$

Differentiating this equation over time, using Eq. (33) for each $\dot{\mathbf{L}}_k$, and again partitioning each force in a accordance with Eq. (29), we get

[18] Here we imply that the internal motions of the particle, including its rotation about its own axis, are negligible. (Otherwise it could not be represented by a geometrical point, as was postulated in Sec. 1.) For a body with substantial rotation (see Chapter 6 below), vector $\mathbf{L}$ retains its definition (32), but is only a part of the total angular momentum and is called the *orbital momentum* – even if the particle does not move along a closed orbit.

[19] See, e.g., MA Eq. (7.3).

$$\dot{\mathbf{L}} = \sum_{\substack{k,k'=1 \\ k'\neq k}}^{N} \mathbf{r}_k \times \mathbf{F}_{kk'} + \boldsymbol{\tau}^{(ext)}, \quad \text{where } \boldsymbol{\tau}^{(ext)} \equiv \sum_{k=1}^{N} \mathbf{r}_k \times \mathbf{F}_k^{(ext)}. \tag{1.37}$$

The first (double) sum may be always divided into pairs of the type ($\mathbf{r}_k \times \mathbf{F}_{kk'} + \mathbf{r}_{k'} \times \mathbf{F}_{k'k}$). With a natural assumption of the central forces ($\mathbf{F}_{kk'} \parallel \mathbf{r}_k - \mathbf{r}_{k'}$), each of these pairs equals zero. Indeed, in this case both components of the pair are vectors perpendicular to the plane passed through positions of both particles and the reference frame origin, i.e. to the plane of drawing of Fig. 4. Also, due to the 3rd Newton law (13) the two forces are equal and opposite, and the magnitude of each term in the sum may be presented as $|F_{kk'}| h_{kk'}$, with equal "lever arms" $h_{kk'} = h_{k'k}$. As a result, each sum ($\mathbf{r}_k \times \mathbf{F}_{kk'} + \mathbf{r}_{k'} \times \mathbf{F}_{k'k}$), and hence the whole double sum in Eq. (37) vanish, and it is reduced to a very simple result,

System' s angular momentum evolution

$$\dot{\mathbf{L}} = \boldsymbol{\tau}^{(\text{ext})}, \tag{1.38}$$

that is similar to Eq. (33) for a single particle, and is the angular analog of Eq. (30). In particular, Eq. (38) shows that if the full external torque $\boldsymbol{\tau}^{(\text{ext})}$ vanishes by some reason (e.g., if the system of particles is isolated from the rest of the Universe), the conservation law (35) is valid for the full angular momentum $\mathbf{L}$, even if its individual components $\mathbf{L}_k$ are not conserved due to inter-particle interactions.

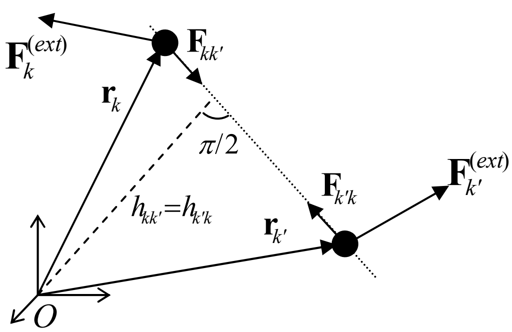

Fig. 1.4. Internal and external forces, and the internal torque cancellation in a system of two particles.

From the mathematical point of view, most conservation laws present *the first integrals of motion* which sometimes liberate us from the necessity to integrate the second-order differential equations of motion, following from the Newton laws, twice.

1.5. Potential energy and equilibrium

Another important role of the potential energy U, especially for dissipative systems whose total mechanical energy E is *not* conserved because it may be drained to the environment, is finding the positions of equilibrium (sometimes called the *fixed points* of the system under analysis) and analyzing their stability with respect to small perturbations. For a single particle, this is very simple: force (22) vanishes at each extremum (minimum or maximum) of the potential energy.[20] Of those fixed points, only the minimums of $U(\mathbf{r})$ are stable – see Sec. 3.2 below for a discussion of this point.

[20] Assuming that the additional, non-conservative forces (such as viscosity) responsible for the mechanical energy drain, vanish at equilibrium – as they typically do. (Static friction is one counter-example.)

A slightly more subtle case is a particle with potential energy $U(\mathbf{r})$, subjected to an *additional* external force $\mathbf{F}^{(\text{ext})}(\mathbf{r})$. In this case, the stable equilibrium is reached at the minimum of not function $U(\mathbf{r})$, but of what is sometimes called the *Gibbs potential energy*

Gibbs' potential energy

$$U_G(\mathbf{r}) \equiv U(\mathbf{r}) - \int^{\mathbf{r}} \mathbf{F}^{(\text{ext})}(\mathbf{r}') \cdot d\mathbf{r}', \tag{1.39}$$

which is defined, just as $U(\mathbf{r})$ is, to an arbitrary constant. The proof of Eq. (39) is very simple: in an extremum of this function, the total force acting on the particle,

$$\mathbf{F}^{(\text{tot})} = \mathbf{F} + \mathbf{F}^{(\text{ext})} \equiv -\nabla U + \nabla \int^{\mathbf{r}} \mathbf{F}^{(\text{ext})}(\mathbf{r}') \cdot d\mathbf{r}' = -\nabla U_G, \tag{1.40}$$

vanishes, as it should.[21] For the simplest (and very frequent) case of the applied force independent on particle's position, the Gibbs potential energy is just

$$U_G(\mathbf{r}) \equiv U(\mathbf{r}) - \mathbf{F}^{(\text{ext})} \cdot \mathbf{r} + \text{const}. \tag{1.41}$$

This is all very straightforward, but since the notion of U_G is not well known to some students,[22] let me offer a very simple example. Consider a 1D deformation of the usual elastic spring providing the returning force ($-\kappa x$), where x is the deviation from spring's equilibrium. In order for the force to comply with Eq. (22), its potential energy should equal to $U = \kappa x^2/2$ + const, so that its minimum corresponds to $x = 0$. This works fine until the spring comes under effect of a nonvanishing external force F, say independent of x. Then the equilibrium deformation of the spring, $x_0 = F/\kappa$, evidently corresponds not to the minimum of U but rather to that of the Gibbs potential energy (41): $U_G = U - Fx = \kappa x^2/2 - Fx$ + const.

1.6. OK, we've got it - can we go home now?

Not yet. In many cases the conservation laws discussed above provide little help, even in systems without dissipation. Consider for example a generalization of the bead-on-the-ring problem shown in Fig. 3, in which the ring is rotated by external forces, with a constant angular velocity ω, about its vertical diameter (Fig. 5).[23] In this problem (to which I will repeatedly return below, using it as

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[21] Physically, the difference $U_G - U$ specified by Eq. (39) may be considered the $\mathbf{r}$-dependent part of the potential energy $U^{(\text{ext})}$ of the external system responsible for the force $\mathbf{F}^{(\text{ext})}$, so that U_G is just the total potential energy $U + U^{(\text{ext})}$, besides the part of $U^{(\text{ext})}$ which does not depend on $\mathbf{r}$ and hence is irrelevant for the fixed point analysis. According to the 3rd Newton law, the force exerted by the particle on the external system equals ($-\mathbf{F}^{(\text{ext})}$), so that its work (and hence the change of $U^{(\text{ext})}$ due to the change of $\mathbf{r}$) is given by the second term in the right-hand part of Eq. (39). Thus the condition of equilibrium, $-\nabla U_G = 0$, is just the condition of an extremum of the total potential energy, $U + U^{(ext)}$, of the two interacting systems.

[22] Unfortunately, in most physics teaching plans the introduction of U_G is postponed until a course of statistical mechanics and/or thermodynamics - where it is a part of the *Gibbs free energy*, in contrast to U, which is a part of the *Helmholtz free energy* - see, e.g., Sec. 1.4 of the *Statistical Mechanics* ("SM") part of my notes. However, the reader should agree that the difference between U_G and U, and hence that between the Gibbs and Helmholtz free energies, has nothing to do with statistics or thermal motion, and belongs to the basic mechanics.

[23] This is essentially a simplified model of the famous mechanical control device called the *centrifugal* (or "flyball, or "centrifugal flyball") *governor* – see, e.g., http://en.wikipedia.org/wiki/Centrifugal_governor.

an analytical mechanics "testbed"), none of the three conservation laws listed in the last section, holds. In particular, bead's energy,

$$E = \frac{m}{2} v^2 + mgh, \tag{1.42}$$

is *not* constant, because the external forces rotating the ring may change it. Of course, we still can solve the problem using the Newton laws, but this is even more complex than for the above case of the ring at rest, in particular because the force **N** exerted on the bead by the ring now may have three rather than two Cartesian components, which are not simply related. One can readily see that if we could exclude the so-called *reaction forces* such as **N**, that ensure *external constraints* of the particle motion, in advance, that would help a lot. Such an exclusion may be provided by analytical mechanics, in particular its Lagrangian formulation, which will be discussed in the next chapter.

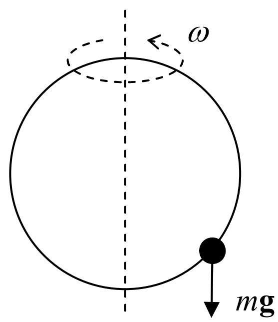


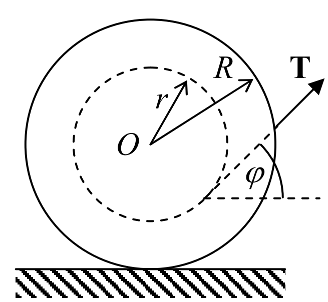
Fig. 1.5. Bead sliding along a rotating ring.

An even more important motivation for analytical mechanics is given by dynamics of "non-mechanical" systems, for example, of the electromagnetic field – possibly interacting with charged particles, conducting bodies, etc. In many such systems, the easiest (and sometimes the only practicable) way to find the equations of motion is to derive then from the Lagrangian or Hamiltonian function of the system. In particular, the Hamiltonian formulation of the analytical mechanics (to be discussed in Chapter 10) offers a direct pathway to deriving Hamiltonian operators of systems, which is the standard entry point for analysis of their quantum-mechanical properties.

1.7. Self-test problems

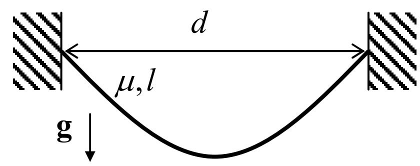
1.1. A bicycle, ridden with velocity v on a wet pavement, has no mudguards on its wheels. How far behind should the following biker ride to avoid being splashed over? Neglect the air resistance effects.

1.2. Two round disks of radius R are firmly connected with a coaxial cylinder of a smaller radius r, and a thread is wound on the resulting spool. The spool is placed on a horizontal surface, and thread's end is being pooled out at angle φ - see Fig. on the right. Assuming that the spool does not slip on the surface, what direction would it roll?

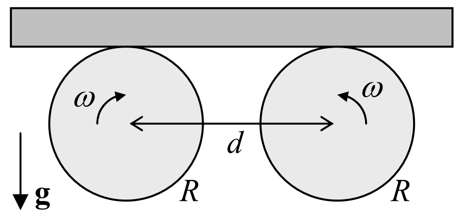


(Sometimes the device is called the "Watt's governor", after the famous engineer J. Watts who used it in 1788 in one of his first steam engines, though it had been used in European windmills at least since the 1600s.)

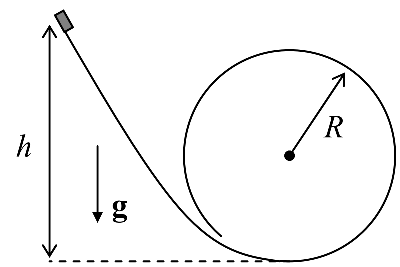
1.3. Calculate the equilibrium shape of a flexible, heavy rope of length l, with a constant mass μ per unit length, if it is hung in a uniform gravity field between two points separated by a horizontal distance d – see Fig. on the right.



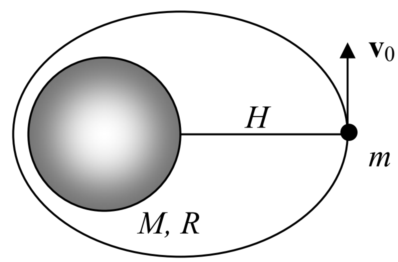
1.4. A uniform, long, thin bar is placed horizontally on two similar round cylinders rotating toward each other with the same angular velocity ω and displaced by distance d – see Fig. on the right. Calculate the laws of relatively slow horizontal motions of the bar within the plane of drawing for both possible directions of cylinder rotation, assuming that the friction force between the slipping surfaces of the bar and each cylinder obeys the usual simple law $|F| = \mu N$, where N is the normal pressure force between them, and μ is a constant (velocity-independent) coefficient. Formulate the condition of validity of your result.



1.5. A small block slides, without friction, down a smooth slide that ends with a round loop of radius R – see Fig. on the right. What smallest initial height h allows the block to make its way around the loop without dropping from the slide, if it is launched with negligible initial velocity?

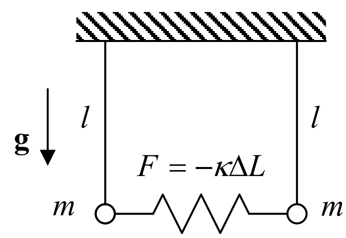


1.6. A satellite of mass m is being launched from height H over the surface of a spherical planet with radius R and mass $M >> m$ - see Fig. on the right. Find the range of initial velocities $\mathbf{v}_0$ (normal to the radius) providing closed orbits above the planet's surface.



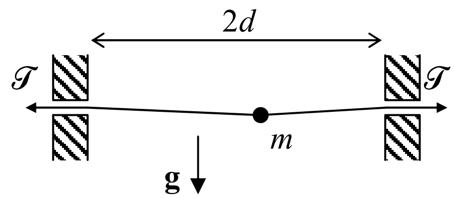
1.7. Prove that the thin-uniform-disk model of a galaxy describes small harmonic oscillations of stars inside it along the direction normal to the disk, and calculate the frequency of these oscillations in terms of the Newton's gravitational constant G and the average density ρ of the star/dust matter of the galaxy.

1.8. Derive the differential equations of motion for small oscillations of two similar pendula coupled with a spring (see Fig. on the right), within the vertical plane. Assume that at the vertical position of both pendula, the spring is not stretched ($\Delta L = 0$).



1.9. One of popular futuristic concepts of travel is digging a straight railway tunnel through the Earth and letting a train go through it, without initial velocity - driven only by gravity. Calculate train's travel time through such a tunnel, assuming that the Earth's density ρ is constant, and neglecting the friction and planet rotation effects.

1.10. A small bead of mass m may slide, without friction, along a light string, stretched with a force $\mathscr{T} >> mg$ between two points separated by a horizontal distance $2d$ – see Fig. on the right. Calculate the frequency of horizontal oscillations of the bead about its equilibrium position.



1.11. Find the acceleration of a rocket due to the working jet motor, and explore the resulting equation of rocket's motion.

Hint: For the sake of simplicity, you may consider a 1D motion.

1.12. Prove the following *virial theorem*:[24] for a set of N particles performing a periodic motion,

$$\overline{T} = -\frac{1}{2}\sum_{k=1}^{N}\overline{\mathbf{F}_k \cdot \mathbf{r}_k},$$

where (as everywhere in these notes), the top bar means time averaging – in this case over the motion period. What does the virial theorem say about:

(i) the 1D motion of a particle in a confining potential $U(x) = ax^{2s}$, with $a > 0$ and $s > 0$, and
(ii) the orbital motion of a particle moving in a central potential $U(r) = - C/r$?

Hint: Explore the time derivative of the following scalar function of time: $G(t) \equiv \sum_{k=1}^{N}\mathbf{p}_k \cdot \mathbf{r}_k$.

[24] It was first stated by R. Clausius in 1870.