A Brief History of Matrices and Linear Algebra

Matrices and linear algebra did not grow out of the study of coefficients of systems of linear equations, as one might guess. Arrays of coefficients led mathematicians to develop determinants, not matrices. Leibniz, coinventor of calculus, used determinants in 1693, about one hundred and fifty years before the study of matrices in their own right. Cramer presented his determinant-based formula for solving systems of linear equations in 1750, and Gauss developed Gaussian elimination around 1820. These events occurred before matrix notation even existed. As an aside, we note that Gaussian elimination was for years considered part of the development of geodesy, not mathematics; the Gauss–Jordan method, which we called elimination by pivoting, first appeared in a handbook on geodesy.

For matrix algebra to develop, one needed two things: (i) the proper notation, such as a_{ij} and $\mathbf{A}$; and (ii) the definition of matrix multiplication. It is interesting that both of these critical factors occurred at about the same time, around 1850, and in the same country, England. Except for Newton's invention of calculus, the major mathematical advances in the seventeenth, eighteenth, and early nineteenth centuries were all made by continental mathematicians, names such as Bernoulli, Cauchy, Euler, Gauss, and Laplace. But in the mid-nineteenth century, English mathematicians pioneered the study of the underlying structure of various algebraic systems. For example, Augustus DeMorgan and George Boole developed the algebra of sets (Boolean algebra) in which symbols were used for propositions and abstract elements.

The introduction of matrix notation and the invention of the word "matrix" were motivated by attempts to develop the right algebraic language for studying determinants. In 1848, J. J. Sylvester introduced the term "matrix," the Latin word for "womb," as a name for an array of numbers. He used "womb" because he viewed a matrix as a generator of determinants. That is, every subset of k rows and k columns in a matrix generated a determinant (associated with the submatrix formed by those rows and columns).

In search of good notation for working with determinants, Sylvester in 1851 proposed writing a square matrix in the form

$$\begin{matrix} a_1\alpha_1 & a_1\alpha_2 & \cdots & a_1\alpha_n \\ a_2\alpha_1 & a_2\alpha_2 & \cdots & a_2\alpha_n \\ \vdots & \vdots & \vdots & \vdots \\ a_n\alpha_1 & a_n\alpha_2 & \cdots & a_n\alpha_n \end{matrix} \tag{1}$$

with each entry represented by a product of symbols. He also introduced the shorthand notation for a square matrix of

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix} \tag{2}$$

He referred to the a's and α's as *umbrae*, or ideal elements. Using this umbral notation, Sylvester then wrote the determinant of (2), which involves summing the signed products of all permutations of the a's with the α's, as

$$\begin{Bmatrix} a_1 & a_2 & \cdots & a_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{Bmatrix} \tag{3}$$

Soon after the introduction of (1), the two symbols a and α were merged into one with double subscripts—a_{ij} (Cauchy had actually used a_{ij} in 1812, but the notation was not accepted then).

Matrix algebra grew out of work by Arthur Cayley in 1855 on linear transformations. Given transformations,

$$T_1: \quad \begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \qquad T_2: \quad \begin{aligned} x'' &= \alpha x' + \beta y' \\ y'' &= \gamma x' + \delta y' \end{aligned}$$

he considered the transformation obtained by performing T_1 and then performing T_2.

$$T_2T_1: \quad \begin{aligned} x'' &= (a\alpha + b\gamma)x + (a\beta + b\delta)y \\ y' &= (c\alpha + d\gamma)x + (c\beta + d\delta)y \end{aligned}$$

In studying ways to represent this composite transformation, he was led to define matrix multiplication: The matrix of coefficients for the composite transformation T_2T_1 is the product of the matrix for T_2 times the matrix for T_1. Cayley went on to study the algebra of these compositions—matrix algebra—including matrix inverses. The use of a single symbol **A** to represent the matrix of a transformation was essential notation of this new algebra. A link between matrix algebra and determinants was quickly established with the result: det(**AB**) = det(**A**) det(**B**). But Cayley believed that matrix algebra would grow to overshadow the theory of determinants. He wrote, "There would be many things to say about this theory of matrices which should, it seems to me, precede the theory of determinants."

It is a curious sidelight to this discussion that another prominent English mathematician of this time was Charles Babbage, who built the first modern calculating machine. Abstracting the mechanics of computation as well as its algebraic structure and notation seems to have been all part of the same general intellectual development in mathematics at that time.

Mathematicians also tried to develop an algebra of vectors, but there was no natural definition for the product of two vectors. The first vector algebra, involving a noncommutative vector product, was proposed by Hermann Grassmann in 1844. Later, Grassmann introduced what we called simple matrices, formed by a column vector times a row vector.

Matrices remained closely associated with linear transformations and, from the theoretical viewpoint, were by 1900 just a finite-dimensional subcase of an emerging general theory of linear transformations. Matrices were also viewed as a powerful notation, but after an initial spurt of interest, were little studied in their own right. More attention was paid to vectors, which are basic mathematical elements of physics as well as many areas of mathematics. The modern definition of a vector space was introduced by Peano in 1888. Abstract vector spaces, whose elements were functions or even linear transformations, soon followed.

Interest in matrices, with emphasis on their numerical analysis, reemerged after World War II with the development of modern digital computers. Von Neumann and Goldstein in 1947 introduced condition numbers in analyzing roundoff error. Alan Turing, the other giant (with von Neumann) in the development of stored-program computers, gave the **LU** decomposition of a matrix in 1948. The usefulness of the **QR** decomposition was realized a decade later.

References

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