Appendix MA

Selected Mathematical Formulas

that are used in this lecture course series,
but not always remembered by students (and some instructors :-)

1. Constants

- Euclidean circle’s length-to-diameter ratio:

\[ \pi = 3.141592653...; \quad \pi^{1/2} \approx 1.77. \]  

(1.1)

- Natural logarithm base:

\[ e \equiv \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n = 2.718281828... \] ;

(1.2a)

from that value, the logarithm base conversion factors are as follows (\( \xi > 0 \)):

\[ \frac{\ln \xi}{\log_{10} \xi} = \ln 10 \approx 2.303, \quad \frac{\log_{10} \xi}{\ln \xi} = \frac{1}{\ln 10} \approx 0.434. \]  

(1.2b)

- The Euler (or “Euler-Mascheroni”) constant:

\[ \gamma \equiv \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln n \right) = 0.5771566490...; \quad e^\gamma \approx 1.781. \]  

(1.3)

2. Combinatorics, sums, and series

(i) Combinatorics

- The number of different permutations, i.e. ordered sequences of \( k \) elements selected from a set of \( n \) distinct elements (\( n \geq k \)), is

\[ \overset{n}{\text{P}}_k = n \cdot (n - 1) \cdot ... \cdot (n - k + 1) = \frac{n!}{(n - k)!}; \]  

(2.1a)

in particular, the number of different permutations of all elements of the set (\( n = k \)) is

\[ \overset{k}{\text{P}}_k = k \cdot (k - 1) \cdot ... \cdot 2 \cdot 1 = k! . \]  

(2.1b)
- The number of different combinations, i.e. unordered sequences of \( k \) elements from a set of \( n \geq k \) distinct elements, is equal to the binomial coefficient

\[
^nC_k = \binom{n}{k} \equiv \frac{n!}{k!(n-k)!}.
\]  
(2.2)

In an alternative, very popular “ball/box language”, \(^nC_k\) is the number of different ways to put in a box, in an arbitrary order, \( k \) balls selected from \( n \) distinct balls.

- A generalization of the binomial coefficient notion is the multinomial coefficient,

\[
^nC_{k_1,k_2,...,k_l} \equiv \frac{n!}{k_1!k_2!...k_l!}, \text{ with } n = \sum_{j=1}^{l} k_j,
\]  
(2.3)

which, in the standard mathematical language, is a number of different permutations in a multiset of \( l \) distinct element types from an \( n \)-element set which contains \( k_j (j = 1, 2, ..., l) \) elements of each type. In the “ball/box language”, coefficient (2.3) is the number of different ways to distribute \( n \) balls between \( l \) different boxes, each time keeping the number \((k_i)\) of balls in the \( j \)-th box fixed, but ignoring their order inside the box. The binomial coefficient \(^nC_k\) (2.2), is a particular case of the multinomial coefficient (2.3) for \( l = 2 \) - counting the explicit box for the first, and the remaining space for the second box, so that if \( k_1 \equiv k \), then \( k_2 = n - k \).

- One more important combinatorial quantity is the number \( M_n^{(k)} \) of ways to place \( n \) indistinguishable balls into \( k \) distinct boxes. It may be readily calculated from Eq. (2.2) as the number of different ways to select \((k-1)\) partitions between the boxes in an imagined linear row of \((k-1+n)\) “objects” (balls in the boxes and partitions between them):

\[
M_n^{(k)} = (-1)^{k-1}C_{k-1} \equiv \frac{(k-1+n)!}{(k-1)!n!}.
\]  
(2.4)

(ii) Sums and series

- Arithmetic progression:

\[
r + 2r + ... + nr \equiv \sum_{k=1}^{n} kr = \frac{n(n+1)r}{2};
\]  
(2.5a)

in particular, at \( r = 1 \) it is reduced to the sum of \( n \) first natural numbers:

\[
1 + 2 + ... + n \equiv \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.
\]  
(2.5b)

- Sums of squares and cubes of \( n \) first natural numbers:

\[
1^2 + 2^2 + ... + n^2 \equiv \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6};
\]  
(2.6a)

\[
1^3 + 2^3 + ... + n^3 \equiv \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.
\]  
(2.6b)

- The Riemann zeta function:
\[
\zeta(s) \equiv 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k^s};
\]  
(2.7a)

the particular values frequently met in applications are

\[
\zeta\left(\frac{3}{2}\right) \approx 2.612, \quad \zeta(2) = \frac{\pi^2}{6}, \quad \zeta\left(\frac{5}{2}\right) \approx 1.341, \quad \zeta(3) \approx 1.202, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(5) \approx 1.037. \quad (2.7b)
\]

- Finite geometric progression (for real \(\lambda \neq 1\)):

\[
1 + \lambda + \lambda^2 + \ldots + \lambda^{n-1} \equiv \sum_{k=0}^{n-1} \lambda^k = \frac{1 - \lambda^n}{1 - \lambda};
\]  
(2.8a)

in particular, if \(\lambda^2 < 1\), the progression has a finite limit at \(n \to \infty\) (called the geometric series):

\[
\lim_{n \to \infty} \sum_{k=0}^{n-1} \lambda^k = \sum_{k=0}^{\infty} \lambda^k = \frac{1}{1 - \lambda}. \quad (2.8b)
\]

- Binomial sum (or the “binomial theorem”):

\[
(1 + a)^n = \sum_{k=0}^{n} \binom{n}{k} a^k,
\]  
(2.9)

where \(\binom{n}{k}\) are the binomial coefficients defined by Eq. (2.2).

- The Stirling formula:

\[
\lim_{n \to \infty} \ln(n!) = n(\ln n - 1) + \frac{1}{2} \ln(2\pi n) + \frac{1}{12n} - \frac{1}{360n^3} + \ldots; 
\]  
(2.10)

for most applications in physics, the first term\(^1\) is sufficient.

- The Taylor (or “Taylor-Maclaurin”) series: for any infinitely differentiable function \(f(\xi)\):

\[
\lim_{\xi \to 0} f(\xi + \tilde{\xi}) = f(\xi) + \frac{df}{d\xi}(\xi) \tilde{\xi} + \frac{1}{2!} \frac{d^2f}{d\xi^2}(\xi) \tilde{\xi}^2 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{d^k f}{d\xi^k}(\xi) \tilde{\xi}^k; 
\]  
(2.11a)

note that for many functions this series converges only within a limited, sometimes small range of deviations \(\tilde{\xi}\). For a function of several arguments, \(f(\xi_1, \xi_2, \ldots, \xi_N)\), the first terms of the Taylor series are

\[
\lim_{\xi_k \to 0} f(\xi_1 + \tilde{\xi}_1, \xi_2 + \tilde{\xi}_2, \ldots) = f(\xi_1, \xi_2, \ldots) + \sum_{k=1}^{N} \frac{\partial f}{\partial \xi_k}(\xi_1, \xi_2, \ldots) \tilde{\xi}_k + \frac{1}{2!} \sum_{k,k=1}^{N} \frac{\partial^2 f}{\partial \xi_k \partial \xi_k}(\xi_1, \xi_2, \ldots) \tilde{\xi}_k^2 + \ldots (2.11b)
\]

- The Euler-Maclaurin formula, valid for any infinitely differentiable function \(f(\xi)\):

\[
\sum_{k=1}^{n} f(k) = \int_{0}^{n} f(\xi) d\xi + \frac{1}{2} \left[ f(n) - f(0) \right] + \frac{1}{6} \cdot \frac{1}{2!} \left[ \frac{d f}{d \xi} (n) - \frac{d f}{d \xi} (0) \right] \\
+ \frac{1}{30} \cdot \frac{1}{4!} \left[ \frac{d^3 f}{d \xi^3} (n) - \frac{d^3 f}{d \xi^3} (0) \right] + \frac{1}{12} \cdot \frac{1}{6!} \left[ \frac{d^5 f}{d \xi^5} (n) - \frac{d^5 f}{d \xi^5} (0) \right] + \ldots; (2.12a)
\]

\(^1\) Actually, this leading term was derived by A. de Moivre in 1733, before the J. Stirling’s work.
the coefficients participating in this formula are the so-called Bernoulli numbers:\(^2\)

\[
B_1 = \frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0, \quad B_4 = \frac{1}{30}, \quad B_5 = 0, \quad B_6 = \frac{1}{42}, \quad B_7 = 0, \quad B_8 = \frac{1}{30}, \ldots
\] (2.12b)

### 3. Basic trigonometric functions

- **Trigonometric functions of the sum and the difference of two arguments:** \(^3\)
  
  \[
  \cos(a \pm b) = \cos a \cos b \mp \sin a \sin b, \quad (3.1a)
  \]
  
  \[
  \sin(a \pm b) = \sin a \cos b \pm \cos a \sin b. \quad (3.1b)
  \]

- **Sums of two functions of arbitrary arguments:**
  
  \[
  \cos a + \cos b = 2 \cos \frac{a + b}{2} \cos \frac{b - a}{2}, \quad (3.2a)
  \]
  
  \[
  \cos a - \cos b = 2 \sin \frac{a + b}{2} \sin \frac{b - a}{2}, \quad (3.2b)
  \]
  
  \[
  \sin a \pm \sin b = 2 \sin \frac{a \pm b}{2} \cos \frac{\pm b - a}{2}. \quad (3.2c)
  \]

- **Trigonometric function products:**
  
  \[
  2 \cos a \cos b = \cos(a + b) + \cos(a - b), \quad (3.3a)
  \]
  
  \[
  2 \sin a \cos b = \sin(a + b) + \sin(a - b), \quad (3.3b)
  \]
  
  \[
  2 \sin a \sin b = \cos(a - b) - \cos(a + b). \quad (3.3c)
  \]

For the particular case of equal arguments, \(b = a\), these three formulas yield the following expressions for the squares of trigonometric functions, and their product:

\[
\cos^2 a = \frac{1}{2}(1 + \cos 2a), \quad \sin a \cos a = \frac{1}{2}\sin 2a, \quad \sin^2 a = \frac{1}{2}(1 - \cos 2a). \quad (3.3d)
\]

- **Cubes of trigonometric functions:**
  
  \[
  \cos^3 a = \frac{3}{4}\cos a + \frac{1}{4}\cos 3a, \quad \sin^3 a = \frac{3}{4}\sin a - \frac{1}{4}\sin 3a. \quad (3.4)
  \]

- **Trigonometric functions of a complex argument:**
  
  \[
  \sin(a + ib) = \sin a \cosh b + i \cos a \sinh b, \quad (3.5)
  \]
  
  \[
  \cos (a + ib) = \cos a \cosh b - i \sin a \sinh b. \quad (3.5)
  \]

---

\(^2\) Note that definitions of \(B_k\) (or rather their signs and indices) differ even in the most popular handbooks.

\(^3\) I am confident that the reader is quite capable of deriving relations (3.1) by representing exponent in the elementary relation \(e^{(a \pm b)} = e^a e^{\pm b}\) as a sum of its real and imaginary parts, Eqs. (3.3) directly from Eqs. (3.1), and Eqs. (3.2) from Eqs. (3.3) by variable replacement; however, I am still providing these formulas to save his or her time. (Quite a few formulas below are included because of the same reason.)
- Sums of trigonometric functions of $n$ equidistant arguments:

$$
\sum_{k=1}^{n} \left\{ \sin \left( k \xi \right) \cos \left( \frac{n+1}{2} \xi \right) \right\} \sin \left( \frac{n}{2} \xi \right) / \sin \left( \frac{\xi}{2} \right).
$$

(3.6)

## 4. General differentiation

- Full differential of a product of two functions:

$$
d(fg) = (df)g + f(dg).
$$

(4.1)

- Full differential of a function of several independent arguments, $f(\xi_1, \xi_2, \ldots, \xi_n)$:

$$
df = \sum_{k=1}^{n} \frac{\partial f}{\partial \xi_k} d\xi_k.
$$

(4.2)

- Curvature of the Cartesian plot of a 1D function $f(\xi)$:

$$
\kappa \equiv \frac{1}{R} = \frac{\left| \frac{d^2 f}{d\xi^2} \right|}{\left[ 1 + \left( \frac{df}{d\xi} \right)^2 \right]^{3/2}}.
$$

(4.3)

## 5. General integration

- Integration by parts - immediately follows from Eq. (4.1):

$$
\int_{g(A)}^{g(B)} f \, dg = \left. f \right|_{g(A)}^{g(B)} - \int_{f(A)}^{f(B)} g \, df.
$$

(5.1)

- Numerical (approximate) integration of 1D functions: the simplest trapezoidal rule,

$$
\int_{a}^{b} f(\xi) \, d\xi \approx h \left[ f \left( a + \frac{h}{2} \right) + f \left( a + \frac{3h}{2} \right) + \ldots + f \left( b - \frac{h}{2} \right) \right] = h \sum_{n=1}^{N} f \left( a - \frac{h}{2} + nh \right), \quad h \equiv \frac{b-a}{N}.
$$

(5.2)

has relatively low accuracy (error of the order of $(h^3/12)d^2f/d\xi^2$ per step), so that the following Simpson formula,

$$
\int_{a}^{b} f(\xi) \, d\xi \approx \frac{h}{3} \left[ f(a) + 4f(a+h) + 2f(a+2h) + \ldots + 4f(b-h) + f(b) \right], \quad h \equiv \frac{b-a}{2N},
$$

(5.3)

whose error per step scales as $(h^5/180)d^4f/d\xi^4$, is used much more frequently.4

4 Higher-order formulas (e.g., the Bode rule), and other guidance including ready-for-use codes for computer calculations may be found, for example, in the popular reference texts by W. H. Press et al., cited in Sec. 16 below. Besides that, some advanced codes are used as subroutines in the software packages listed in the same section. In some cases, the Euler-Maclaurin formula (2.12) also may be useful for numerical integration.
6. A few 1D integrals

(i) Indefinite integrals

- Integrals with \((1 + \xi^2)^{1/2}\):

\[
\int (1 + \xi^2)^{1/2} d\xi = \frac{\xi}{2} (1 + \xi^2)^{1/2} + \frac{1}{2} \ln \left| \xi + (1 + \xi^2)^{1/2} \right|, \tag{6.1}
\]

\[
\int \frac{d\xi}{(1 + \xi^2)^{1/2}} = \ln \left| \xi + (1 + \xi^2)^{1/2} \right|, \tag{6.2a}
\]

\[
\int \frac{d\xi}{(1 + \xi^2)^{1/2}} = \frac{\xi}{(1 + \xi^2)^{1/2}}. \tag{6.2b}
\]

- Miscellaneous indefinite integrals:

\[
\int \frac{d\xi}{\xi (\xi^2 + 2a\xi - 1)^{1/2}} = \arccos \left( \frac{a\xi - 1}{\xi (a^2 + 1)^{1/2}} \right), \tag{6.3a}
\]

\[
\int \frac{d\xi}{\xi^5 (\sin \xi - \xi \cos \xi)^2} = \frac{2 \xi \sin 2\xi + \cos 2\xi - 2\xi^2 - 1}{8\xi^4}, \tag{6.3b}
\]

\[
\int \frac{d\xi}{a + b \cos \xi} = \frac{2}{(a^2 - b^2)^{1/2}} \tan^{-1} \left( \frac{(a-b)}{(a^2 - b^2)^{1/2}} \tan \frac{\xi}{2} \right), \quad \text{for } a^2 > b^2. \tag{6.3c}
\]

\[
\int \frac{d\xi}{1 + \xi^2} = \tan^{-1} \xi. \tag{6.3d}
\]

(ii) Semi-definite integrals:

- Integrals with \(1/(e^{\xi} \pm 1)\):

\[
\int_{a}^{\infty} \frac{d\xi}{e^{\xi} + 1} = \ln \left( 1 + e^{-a} \right), \tag{6.4a}
\]

\[
\int_{a>0}^{\infty} \frac{d\xi}{e^{\xi} - 1} = \ln \frac{1}{1 - e^{-a}}. \tag{6.4b}
\]

(iii) Definite integrals

- Integrals with \(1/(1 + \xi^2)\):

\[
\int_{0}^{\infty} \frac{d\xi}{1 + \xi^2} = \frac{\pi}{2}, \tag{6.5a}
\]

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5 A powerful (and free :-) interactive online tool for working out indefinite 1D integrals is available at [http://integrals.wolfram.com/index.jsp](http://integrals.wolfram.com/index.jsp).

6 Eq. (6.5a) follows immediately from Eq. (6.3d), and Eq. (6.5b) from Eq. (6.2b) – a couple more examples of the (intentional) redundancy of this list.
\[ \int_0^\infty \frac{d\xi}{(1 + \xi^2)^{3/2}} = 1; \]  
more generally,
\[ \int_0^\infty \frac{d\xi}{(1 + \xi^2)^{2/3}} = \frac{\pi}{2} \left( \frac{2n-3}{2n-2} \right)! \equiv \frac{\pi}{2} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2 \cdot 4 \cdot 6 \cdots (2n-2)}, \quad \text{for } n = 2, 3, \ldots \]  

- Integrals with \((1 - \xi^{2n})^{1/2}\):
\[ \int_0^1 \frac{d\xi}{(1 - \xi^{2n})^{1/2}} = \frac{\pi^{1/2}}{2n} \left( \frac{1}{2n} \right) \Gamma\left( \frac{n+1}{2n} \right), \]  
\[ \int_0^1 (1 - \xi^{2n})^{-1/2} d\xi = \frac{\pi^{1/2}}{4n} \left( \frac{3}{2n} \right) \Gamma\left( \frac{3n+1}{2n} \right), \]  
where \(\Gamma(s)\) is the gamma-function, which is most often defined (for \(\text{Re } s > 0\)) by the following integral:
\[ \int_0^\infty \xi^{s-1} e^{-\xi} d\xi = \Gamma(s). \]  
The key property of this function is the recurrence relation, valid for any \(s \neq 0, -1, -2, \ldots\), is
\[ \Gamma(s+1) = s\Gamma(s). \]  
Since, according to Eq. (6.7a), \(\Gamma(1) = 1\), Eq. (6.7b) for non-negative integers takes the form
\[ \Gamma(n+1) = n!, \quad \text{for } n = 0, 1, 2, \ldots \]  
(\(\text{where } 0! \equiv 1\)). Because of this, for integer \(s = n + 1 \geq 1\), Eq. (6.7a) is reduced to
\[ \int_0^\infty \xi^n e^{-\xi} d\xi = n!. \]  
Other frequently met values of the gamma-function are those for positive semi-integer arguments:
\[ \Gamma\left( \frac{1}{2} \right) = \pi^{1/2}, \quad \Gamma\left( \frac{3}{2} \right) = \frac{1}{2} \pi^{1/2}, \quad \Gamma\left( \frac{5}{2} \right) = \frac{1}{2} \cdot \frac{3}{2} \pi^{1/2}, \quad \Gamma\left( \frac{7}{2} \right) = \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \pi^{1/2}, \ldots \]  

- Integrals with \(1/(e^\xi \pm 1)\):
\[ \int_0^\infty \frac{\xi^{s-1} d\xi}{e^\xi + 1} = \left( 1 - 2^{1-s} \right) \zeta(s), \quad \text{for } s > 0, \]  
\[ \int_0^\infty \frac{\xi^{s-1} d\xi}{e^\xi - 1} = \zeta(s), \quad \text{for } s > 1, \]  
where \(\zeta(s)\) is the Riemann zeta-function – see Eq. (2.6). Particular cases: for \(s = 2n\),
\[ \int_0^\infty \frac{\xi^{2n-1} d\xi}{e^\xi + 1} = \frac{2^{2n-1} - 1}{2n} \pi^{2n} B_{2n}, \]
where $B_n$ are the Bernoulli numbers – see Eq. (2.12). For the particular case $s = 1$ (when Eq. (6.8a) yields uncertainty),

$$\int_0^\infty \frac{d\xi}{e^\xi - 1} = \ln 2 .$$

(6.8e)

- Integrals with $\exp\{-\xi^2\}$:

$$\int_0^\infty \xi^s e^{-\xi^2} d\xi = \frac{1}{2} \Gamma\left(\frac{s + 1}{2}\right), \quad \text{for } s > -1 ;$$

(6.9a)

for applications the most important particular even values of $s$ are 0 and 2:

$$\int_0^\infty e^{-\xi^2} d\xi = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\pi^{1/2}}{2} ,$$

(6.9b)

$$\int_0^\infty \xi^2 e^{-\xi^2} d\xi = \frac{1}{2} \Gamma\left(\frac{3}{2}\right) = \frac{\pi^{1/2}}{4} ,$$

(6.9c)

though we will also run into the cases $s = 4$ and $s = 6$:

$$\int_0^\infty \xi^4 e^{-\xi^2} d\xi = \frac{1}{2} \Gamma\left(\frac{5}{2}\right) = \frac{3\pi^{1/2}}{8} , \quad \int_0^\infty \xi^6 e^{-\xi^2} d\xi = \frac{1}{2} \Gamma\left(\frac{7}{2}\right) = \frac{15\pi^{1/2}}{16} ;$$

(6.9d)

for odd values $s = 2n + 1$ (with $n = 0, 1, 2, \ldots$), Eq. (6.9a) takes a simpler form

$$\int_0^\infty \xi^{2n+1} e^{-\xi^2} d\xi = \frac{1}{2} \Gamma(n + 1) = \frac{n!}{2} .$$

(6.9e)

- Integrals with cosine and sine functions:

$$\int_0^\infty \cos(\xi^2) d\xi = \int_0^\infty \sin(\xi^2) d\xi = \left(\frac{\pi}{8}\right)^{1/2} .$$

(6.10)

$$\int_0^\infty \frac{\cos \xi}{a^2 + \xi^2} d\xi = \frac{\pi}{2a} e^{-a} .$$

(6.11)

$$\int_0^\infty \left(\frac{\sin \xi}{\xi}\right)^2 d\xi = \frac{\pi}{2} .$$

(6.12)

- Integrals with logarithms:

$$\int_0^1 \ln \frac{a + (1 - \xi^2)^{1/2}}{a - (1 - \xi^2)^{1/2}} d\xi = \pi \left[a - (a^2 - 1)^{1/2}\right], \quad \text{for } a \geq 1 .$$

(6.13)

$$\int_0^1 \ln \frac{1 + (1 - \xi^{1/2})^{1/2}}{\xi^{1/2}} d\xi = 1 .$$

(6.14)
- Integral representations of the Bessel functions of integer order:

\[
J_n(\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(\alpha \sin \xi - n\xi)} d\xi, \quad \text{so that} \quad e^{i\alpha \sin \xi} = \sum_{k=-\infty}^{\infty} J_k(\alpha) e^{ik\xi}; \quad (6.15a)
\]

\[
I_n(\alpha) = \frac{1}{\pi} \int_{0}^{\pi} e^{\alpha \cos \xi} \cos n\xi \, d\xi. \quad (6.15b)
\]

7. 3D vector products

(i) Definitions:

- **Scalar** ("dot-") product:

\[
a \cdot b = \sum_{j=1}^{3} a_j b_j, \quad (7.1)
\]

where \(a_j\) and \(b_j\) are vector components in any orthogonal coordinate system. In particular, vector squared (the same as norm squared):

\[
a^2 \equiv a \cdot a = \sum_{j=1}^{3} a_j^2 \equiv \|a\|^2. \quad (7.2)
\]

- **Vector** ("cross-") product:

\[
a \times b = n_1(a_2b_3 - a_3b_2) + n_2(a_3b_1 - a_1b_3) + n_3(a_1b_2 - a_2b_1) = \begin{vmatrix} n_1 & n_2 & n_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \quad (7.3)
\]

where \(\{n_j\}\) is the set of mutually perpendicular unit vectors\(^7\) along the corresponding coordinate system axes.\(^8\) In particular, Eq. (7.3) yields

\[
a \times a = 0. \quad (7.4)
\]

(ii) Corollaries (readily verified by Cartesian components):

- Double vector product (the so-called *bac minus cab* rule):

\[
a \times (b \times c) = b(a \cdot c) - c(a \cdot b). \quad (7.5)
\]

- Mixed scalar-vector product (called the *operand rotation rule*):

\[
a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b). \quad (7.6)
\]

- Scalar product of vector products:

\[
(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c); \quad (7.7a)
\]

in the particular case of two similar operands (say, \(a = c\) and \(b = d\)), the last formula is reduced to

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\(^7\) Other popular notations for this vector set are \(\{\hat{e}_j\}\) and \(\{\hat{r}_j\}\).

\(^8\) It is easy to use Eq. (7.3) to check that the direction of the product vector corresponds to the well-known "right-hand rule" and to the even more convenient *corkscrew rule*: if we rotate a corkscrew's handle from the first operand toward the second one, its axis moves in the direction of the product.
\[(a \times b)^2 = (ab)^2 - (a \cdot b)^2. \] (7.7b)

8. Differentiation in 3D Cartesian coordinates

- Definition of the del (or “nabla”) vector-operator \( \nabla \): \(^9\)

\[
\nabla \equiv \sum_{j=1}^{3} n_j \frac{\partial}{\partial r_j},
\] (8.1)

where \( r_j \) is a set of linear and orthogonal (called Cartesian) coordinates along directions \( n_j \). In accordance with this definition, the operator \( \nabla \) acting on a scalar function of coordinates, \( f(r) \), \(^{10}\) gives its gradient, i.e. a new vector:

\[
\nabla f \equiv \sum_{j=1}^{3} n_j \frac{\partial f}{\partial r_j} = \text{grad } f.
\] (8.2)

- The scalar product of del by a vector function of coordinates (a vector field),

\[
f(r) \equiv \sum_{j=1}^{3} n_j f_j(r),
\] (8.3)

compiled formally following Eq. (7.1), is a scalar function – the divergence of the initial function:

\[
\nabla \cdot f \equiv \sum_{j=1}^{3} \frac{\partial f_j}{\partial r_j} = \text{div } f,
\] (8.4)

while the vector product of \( \nabla \) and \( f \), formed in a formal accordance with Eq. (7.3), is a new vector - the curl (in European tradition, called rotor and denoted rot) of \( f \):

\[
\nabla \times f \equiv \begin{vmatrix}
    n_1 & n_2 & n_3 \\
    \frac{\partial}{\partial r_1} & \frac{\partial}{\partial r_2} & \frac{\partial}{\partial r_3} \\
    f_1 & f_2 & f_3
\end{vmatrix} = n_1 \left( \frac{\partial f_3}{\partial r_2} - \frac{\partial f_2}{\partial r_3} \right) + n_2 \left( \frac{\partial f_1}{\partial r_3} - \frac{\partial f_3}{\partial r_1} \right) + n_3 \left( \frac{\partial f_2}{\partial r_1} - \frac{\partial f_1}{\partial r_2} \right) \equiv \text{curl } f. \] (8.5)

- One more frequently met “product” is \((f \cdot \nabla)g\), where \( f \) and \( g \) are two arbitrary vector functions of \( r \). This product should be also understood in the sense implied by Eq. (7.1), i.e. as a vector whose \( j \)-th Cartesian component is

\[
[(f \cdot \nabla)g] = \sum_{j=1}^{3} f_j \frac{\partial g_j}{\partial r_j}.
\] (8.5)

9. The Laplace operator \( \nabla^2 = \nabla \cdot \nabla \)

- Expression in Cartesian coordinates - in the formal accordance with Eq. (7.2):

\(^9\) One can run into the following notation: \( \nabla \equiv \partial / \partial r \), which is convenient is some cases, but may be misleading in quite a few others, so it will be not used in these notes.

\(^{10}\) In this, and four next sections, all scalar and vector functions are assumed to be differentiable.
- According to its definition, the Laplace operator acting on a scalar function of coordinates gives a new scalar function:

\[ \nabla^2 f \equiv \nabla \cdot (\nabla f) = \text{div} (\text{grad} f) = \sum_{j=1}^{3} \frac{\partial^2 f}{\partial r_j^2}. \]  

(9.2)

- On the other hand, acting on a vector function (8.3), operator \( \nabla^2 \) returns another vector:

\[ \nabla^2 \mathbf{f} = \sum_{j=1}^{3} n_j \nabla^2 f_j. \]  

(9.3)

Note that Eqs. (9.1)-(9.3) are only valid in Cartesian (i.e. orthogonal and linear) coordinates, but generally not in other (even orthogonal) coordinates – see, e.g., Eqs. (10.6) and (10.12) below.

10. Operators \( \nabla \) and \( \nabla^2 \) in the most important systems of orthogonal coordinates\(^{11}\)

(i) **Cylindrical**\(^{12}\) coordinates \{\( \rho, \varphi, z \}\) (see Fig. below) may be defined by their relations with the Cartesian coordinates:

\[
\begin{align*}
 r_3 &= z, \\
 r_1 &= \rho \cos \varphi, \\
 r_2 &= \rho \sin \varphi, \\
 r_3 &= z.
\end{align*}
\]

(10.1)

- Gradient of a scalar function:

\[ \nabla f = n_\rho \frac{\partial f}{\partial \rho} + n_\varphi \frac{1}{\rho} \frac{\partial f}{\partial \varphi} + n_z \frac{\partial f}{\partial z}. \]

(10.2)

- The Laplace operator of a scalar function:

\[ \nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}, \]

(10.3)

- Divergence of a vector function of coordinates (\( \mathbf{f} = n_\rho \rho \mathbf{f}_\rho + n_\varphi \rho \mathbf{f}_\varphi + n_z \mathbf{f}_z \)):

\[ \nabla \cdot \mathbf{f} = \frac{1}{\rho} \frac{\partial (\rho f_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial f_\varphi}{\partial \varphi} + \frac{\partial f_z}{\partial z}. \]

(10.4)

- Curl of a vector function:

\[ \nabla \times \mathbf{f} = \]
\[ \nabla \times \mathbf{f} = n_r \left( \frac{1}{\rho} \frac{\partial f_r}{\partial \varphi} - \frac{\partial f_{\rho}}{\partial z} \right) + n_\varphi \left( \frac{\partial f_r}{\partial z} - \frac{\partial f_{\rho}}{\partial \rho} \right) + n_z \left( \frac{\partial (\rho f_{\rho})}{\partial \rho} - \frac{\partial f_{\varphi}}{\partial \varphi} \right). \]  \hspace{1cm} (10.5)

- The Laplace operator of a vector function:

\[ \nabla^2 \mathbf{f} = n_r \left( \nabla^2 f_r - \frac{1}{\rho^2} f_r - \frac{2}{\rho^2} \frac{\partial f_{\rho}}{\partial \varphi} \right) + n_\varphi \left( \nabla^2 f_{\rho} - \frac{1}{\rho^2} f_{\rho} + \frac{2}{\rho^2} \frac{\partial f_{\varphi}}{\partial \varphi} \right) + n_z \nabla^2 f_z. \]  \hspace{1cm} (10.6)

(ii) Spherical coordinates \( \{ r, \theta, \varphi \} \) (see Fig. below) may be defined as:

\[ r_1 = r \sin \theta \cos \varphi, \]
\[ r_2 = r \sin \theta \sin \varphi, \]
\[ r_3 = r \cos \theta. \]  \hspace{1cm} (10.7)

- Gradient of a scalar function:

\[ \nabla f = n_r \frac{\partial f}{\partial r} + n_\theta \frac{1}{r} \frac{\partial f}{\partial \theta} + n_\varphi \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi}. \]  \hspace{1cm} (10.8)

- The Laplace operator of a scalar function:

\[ \nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{(r \sin \theta)^2} \frac{\partial^2 f}{\partial \varphi^2}. \]  \hspace{1cm} (10.9)

- Divergence of a vector function \( \mathbf{f} = n_r f_r + n_\theta f_\theta + n_\varphi f_\varphi \):

\[ \nabla \cdot \mathbf{f} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 f_r \right) + \frac{1}{r \sin \theta} \frac{\partial (f_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial f_\varphi}{\partial \varphi}. \]  \hspace{1cm} (10.10)

- Curl of a similar vector function:

\[ \nabla \times \mathbf{f} = n_r \frac{1}{r \sin \theta} \left( \frac{\partial (f_\theta \sin \theta)}{\partial \theta} - \frac{\partial f_\varphi}{\partial \varphi} \right) + n_\theta \frac{1}{r} \left( \frac{\partial f_r}{\partial \theta} - \frac{\partial (rf_\theta)}{\partial r} \right) + n_\varphi \frac{1}{r} \left( \frac{\partial (rf_\theta)}{\partial r} - \frac{\partial f_r}{\partial \theta} \right). \]  \hspace{1cm} (10.11)

- The Laplace operator of a vector function:

\[ \nabla^2 \mathbf{f} = n_r \left( \nabla^2 f_r - \frac{2}{r^2} f_r - \frac{2}{r^2 \sin \theta} \frac{\partial f_r}{\partial \theta} - \frac{2}{r^2 \sin \theta} \frac{\partial f_\varphi}{\partial \varphi} \right) + n_\theta \left( \nabla^2 f_\theta - \frac{2}{r^2 \sin \theta} f_\theta + \frac{2}{r^2 \sin \theta} \frac{\partial f_r}{\partial \varphi} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial f_\varphi}{\partial \varphi} \right) \]
\[ + n_\varphi \left( \nabla^2 f_\varphi - \frac{2}{r^2 \sin^2 \theta} f_\varphi + \frac{2}{r^2 \sin \theta} \frac{\partial f_r}{\partial \varphi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial f_\theta}{\partial \varphi} \right). \]  \hspace{1cm} (10.12)
11. Products involving $\nabla$

(i) Useful zeros:
- For any scalar function $f(r)$, 
  \[ \nabla \times (\nabla f) \equiv \text{curl} (\nabla f) = 0. \quad (11.1) \]
- For any vector function $f(r)$, 
  \[ \nabla \cdot (\nabla \times f) \equiv \text{div} (\nabla f) = 0. \quad (11.2) \]

(ii) Laplace operator expressed via the curl of a curl:
  \[ \nabla^2 f = \nabla (\nabla \cdot f) - \nabla \times (\nabla \times f). \quad (11.3) \]

(iii) Spatial differentiation of a product of a scalar function by a vector function:
- The scalar 3D generalization of Eq. (4.1) is 
  \[ \nabla \cdot (fg) = (\nabla f) \cdot g + f(\nabla \cdot g), \quad (11.4a) \]
  and its vector generalization is similar:
  \[ \nabla \times (fg) = (\nabla f) \times g + f(\nabla \times g). \quad (11.4b) \]

(iv) 3D spatial differentiation of products of two vector functions:
  \[ \nabla \times (fg) = (\nabla f) \cdot g - (\nabla g) \cdot f + (\nabla \cdot f)g + (\nabla \cdot g)f, \quad (11.5) \]
  \[ \nabla (fg) = (\nabla f)g + (\nabla g)f + f(\nabla \times g) + g(\nabla \times f), \quad (11.6) \]
  \[ \nabla \cdot (fg) = g(\nabla \times f) - f(\nabla \times g). \quad (11.7) \]

12. Integro-differential relations

(i) For an arbitrary surface $S$ limited by closed contour $C$:
- The **Stokes theorem**, valid for any differentiable vector field $f(r)$:
  \[ \int_S (\nabla \times f) \cdot d^2 r = \oint_C (\nabla \times f) \cdot d r = \oint_C f \cdot d r \equiv \oint_C f, dr, \quad (12.1) \]
  where $d^2 r \equiv n d^2 r$ is the elementary area vector (normal to the surface), and $d r$ is the elementary contour length vector (tangential to the contour line).

(ii) For an arbitrary volume $V$ limited by closed surface $S$:
- **Divergence** (or “Gauss”) **theorem**, valid for any differentiable vector field $f(r)$:
  \[ \int_V (\nabla \cdot f) d^3 r = \oint_S f \cdot d^3 r \equiv \int_S f, d^3 r, \quad (12.2) \]
- **Green’s theorem**, valid for two differentiable scalar functions $f(r)$ and $g(r)$:
  \[ \int_V \left(f \nabla^2 g - g \nabla^2 f\right) d^3 r = \oint_S \left(f \nabla g - g \nabla f\right) \cdot d^2 r. \quad (12.3) \]
- An identity valid for any two scalar functions \( f \) and \( g \), and a vector field \( j \) with \( \nabla \cdot j = 0 \) (all differentiable):
\[
\int_{\mathcal{V}} \left[ f(\mathbf{j} \cdot \nabla g) + g(\mathbf{j} \cdot \nabla f) \right] d^3 r = \oint_{\mathcal{S}} fgj_\alpha d^2 r.
\] (12.3)

13. The Kronecker delta and Levi-Civita permutation symbols
- The Kronecker delta symbol (defined for integer indices):
\[
\delta_{jj'} = \begin{cases} 
1, & \text{if } j' = j, \\
0, & \text{otherwise}.
\end{cases}
\] (13.1)

- The Levi-Civita permutation symbol (most frequently used for 3 integer indices, each taking one of values 1, 2, or 3):
\[
\varepsilon_{jj'k'} = \begin{cases} 
+1, & \text{if the indices follow in the "correct" ("even") order:} 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow 2, \ldots, \\
-1, & \text{if the indices follow in the "incorrect" ("odd") order:} 1 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 3, \ldots, \\
0, & \text{if any two indices coincide.}
\end{cases}
\] (13.2)

- Relation between the Levi-Civita and the Kronecker delta products:
\[
\varepsilon_{jj'k'}\varepsilon_{kk'r'} = \sum_{l,l',l''=1}^3 \left| \begin{array}{ccc} 
\delta_{jl} & \delta_{jl'} & \delta_{jl''} \\
\delta_{j'l} & \delta_{j'l'} & \delta_{j'l''} \\
\delta_{j''l} & \delta_{j''l'} & \delta_{j''l''}
\end{array} \right| ;
\] (13.3a)

summation of this relation, written for 3 different values of \( j = k \), over these values yields the so-called contracted epsilon identity:
\[
\sum_{j=1}^3 \varepsilon_{jj'k'}\varepsilon_{kk'r'} = \delta_{jk}\delta_{jr'} - \delta_{jr}\delta_{jk'}.
\] (13.3b)

14. Dirac’s delta-function, sign function, and theta-function
- Definition of 1D delta-function (for real \( a < b \)):
\[
\int_a^b f(\xi)\delta(\xi)d\xi = \begin{cases} 
f(0), & \text{if } a < 0 < b, \\
0, & \text{otherwise},
\end{cases}
\] (14.1)

where \( f(\xi) \) is any function continuous near \( \xi = 0 \). In particular (if \( f(\xi) = 1 \) near \( \xi = 0 \)), the definition yields
\[
\int_a^b \delta(\xi)d\xi = \begin{cases} 
1, & \text{if } a < 0 < b, \\
0, & \text{otherwise}.
\end{cases}
\] (14.2)

- Relation to the theta-function \( \theta(\xi) \) and sign function \( \text{sgn}(\xi) \)
\[ \delta(\xi) = \frac{d}{d\xi} \theta(\xi) = \frac{1}{2} \frac{d}{d\xi} \text{sgn}(\xi), \]  

(14.3a)

where

\[ \theta(\xi) \equiv \frac{\text{sgn}(\xi) + 1}{2} = \begin{cases} 0, & \text{if } \xi < 0, \\ 1, & \text{if } \xi > 1, \end{cases} \quad \text{sgn}(\xi) \equiv \frac{\xi}{|\xi|} = \begin{cases} -1, & \text{if } \xi < 0, \\ +1, & \text{if } \xi > 1. \end{cases} \]  

(14.3b)

- An important integral:

\[ \int_{-\infty}^{\infty} e^{is\xi} \, ds = 2\pi \delta(\xi). \]  

(14.4)

- 3D generalization of the delta-function of the radius-vector (the 2D generalization is similar):

\[ \int_{V} f(\mathbf{r}) \delta(\mathbf{r}) \, d^3r = \begin{cases} f(0), & \text{if } 0 \in V, \\ 0, & \text{otherwise}; \end{cases} \]  

(14.5)

it may be presented as a product of 1D delta-functions of Cartesian coordinates:

\[ \delta(\mathbf{r}) = \delta(r_1) \delta(r_2) \delta(r_3). \]  

(14.6)

15. The Cauchy theorem and integral

Let a complex function \( f(z) \) be analytic within a part of the complex plane \( z \), that is limited by a closed contour \( C \) and includes point \( z' \). Then

\[ \oint_{C} f(z) \, dz = 0, \]  

(15.1)

\[ \oint_{C} \frac{dz}{z - z'} = 2\pi i f(z'). \]  

(15.2)

The first of these relations is usually called the Cauchy integral theorem (or the “Cauchy-Goursat theorem”), and the second one - the Cauchy integral (or the “Cauchy integral formula”).

16. References

(i) Properties of some special functions are briefly discussed at the relevant points of the lecture notes; in the alphabetical order:

- Airy functions: QM Sec. 2.4;
- Bessel functions: EM Sec. 2.7;
- Fresnel integrals: EM Sec. 8.6;

\[ f(s) \equiv 1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-is\xi} \left[ 2\pi \delta(\xi) \right] d\xi. \]

\[ 13 \] The coefficient in this equation may be readily recalled by considering its left-hand part as the Fourier-integral presentation of function \( f(s) \equiv 1 \), and applying Eq. (14.1) to the reciprocal Fourier transform
- Hermite polynomials: QM Sec. 2.9;
- Laguerre polynomials (both simple and associated): QM Sec. 3.7;
- Legendre polynomials, associated Legendre functions: EM Sec. 2.8 and QM Sec. 3.6;
- Spherical harmonics: QM Sec. 3.6;
- Spherical Bessel functions: QM Secs. 3.6 and 3.8.

(ii) For more formulas, and their discussions, I can recommend the following handbooks (in the alphabetical order):\(^\text{14}\)

- M. Abramowitz and I. Stegun (eds.), *Handbook of Mathematical Formulas*, Dover, 1965 (and numerous later printings);\(^\text{15}\)

A popular textbook,


may be also used as a formula manual.

Many formulas are also available from the symbolic calculation parts of commercially available software packages listed in Sec. (iv) below.

(iii) Probably the most popular collection of numerical calculation codes are the twin manuals


My lecture notes include very brief introductions into numerical methods of differential equation solution:

- ordinary differential equations: CM Sec. 3.9, and
- partial differential equations: CM Sec. 8.5 and EM Sec. 2.8,

which include references to literature for further reading.

(iv) The most popular software packages for numerical and symbolic calculations, all with plotting capabilities (in the alphabetical order):

- Maple (http://www.maplesoft.com/);
- MathCAD (http://www.ptc.com/products/matcad/);
- Mathematica (http://www.wolfram.com/products/mathematica/index.html);
- MATLAB (http://www.mathworks.com/products/matlab/).

\(^{14}\) On a personal note, perhaps 90\% of all formula needs throughout my research career were satisfied by a tiny, wonderfully compiled old book: H. Dwight, *Tables of Integrals and Other Mathematical Formulas*, 4\(^{\text{th}}\) ed., Macmillan, 1961, whose used copies, rather amazingly, are still available on the Web.

\(^{15}\) An updated version of this collection is now available online at http://dlmf.nist.gov/.