Chapter 8. Radiation, Scattering, Interference, and Diffraction

This chapter continues the discussion of the electromagnetic wave propagation, now focusing on the results of wave incidence on a passive object. Depending on the object’s shape, the result of this interaction is called either scattering, or diffraction, or interference. However, as we will see below, the boundary between these effects is blurry, and their mathematical description may be conveniently based on a single key calculation - the electric dipole radiation of a spherical wave by a small source. Naturally, I will start the chapter from this calculation, deriving it from an even more general result – the “retarded potentials” solution of the Maxwell equations.

8.1. Retarded potentials

Let us start from the general solution of the Maxwell equations in a dispersion-free, linear, uniform, isotropic medium, characterized by frequency-independent, real \( \varepsilon \) and \( \mu \) - for example, free space. The easiest way to perform this calculation is to use the scalar (\( \phi \)) and vector (\( \mathbf{A} \)) potentials of electromagnetic field, that are defined via the electric and magnetic fields by Eqs. (6.106):

\[
\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}.
\]  

As was discussed in Chapter 6, imposing upon the potentials the Lorenz gauge condition (6.108),

\[
\nabla \cdot \mathbf{A} + \frac{1}{v^2} \frac{\partial \phi}{\partial t} = 0, \quad v^2 = \frac{1}{\varepsilon \mu},
\]  

(which does not affect fields \( \mathbf{E} \) and \( \mathbf{B} \)) the macroscopic Maxwell equations for the fields may be recast into a pair of very similar, simple equations (6.109) for the potentials:

\[
\nabla^2 \phi - \frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\varepsilon},
\]

\[
\nabla^2 \mathbf{A} - \frac{1}{v^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{j}. \tag{8.3b}
\]

Let us calculate the fields induced by the stand-alone electric charge and current densities \( \rho(\mathbf{r}, t) \) and \( \mathbf{j}(\mathbf{r}, t) \), thinking of them as known functions. The idea how this may be done may be borrowed from electro- and magnetostatics. Indeed, for the stationary case (\( \partial / \partial t = 0 \)), the solutions of Eqs. (8.3) are given, by the evident generalization of, respectively, Eq. (1.38) and by Eq. (5.28) to the uniform, linear medium:

\[
\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon} \int \rho(\mathbf{r}') \frac{d^3 r'}{|\mathbf{r} - \mathbf{r}'|}, \tag{8.4a}
\]

\[
\mathbf{A}(\mathbf{r}) = \frac{1}{\mu_0} \int \mathbf{j}(\mathbf{r}') \frac{d^3 r'}{|\mathbf{r} - \mathbf{r}'|}.
\]

1 When necessary (e.g., at the discussion of the Cherenkov radiation in Sec. 10.4), it will be not too hard to generalize these results to dispersive media.

2 Such thinking would not prevent the results from being valid for the case when \( \rho(\mathbf{r}, t) \) and \( \mathbf{j}(\mathbf{r}, t) \) should be calculated self-consistently.
\[ \mathbf{A}(\mathbf{r}) \equiv \frac{\mu}{4\pi} \int \mathbf{j}(\mathbf{r}') \frac{d^3r'}{|\mathbf{r} - \mathbf{r}'|}. \] (8.4b)

As we know, these expressions may be derived by, first, calculating the potential of a point source, and then using the linear superposition principle for a system of such sources.

Let us do the same for the time-dependent case, starting from the field induced by a time-dependent point charge at origin:\(^3\)

\[ \rho(\mathbf{r}, t) = q(t)\delta(\mathbf{r}), \] (8.5)

In this case Eq. (3a) is homogeneous everywhere but the origin:

\[ \nabla^2 \phi - \frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2} = 0, \quad \text{at } r \neq 0. \] (8.6)

Due to the spherical symmetry of the problem, it is natural to look for a spherically-symmetric solution to this equation.\(^4\) Thus, we may simplify the Laplace operator\(^5\) correspondingly, and reduce Eq. (6) to

\[ \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right] \phi = 0, \quad \text{at } r \neq 0. \] (8.7)

If we now introduce a new variable \( \chi \equiv r\phi \), Eq. (7) is reduced to the 1D wave equation

\[ \left( \frac{\partial^2}{\partial r^2} - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right) \chi = 0, \quad \text{at } r \neq 0. \] (8.8)

From the discussion in Chapter 7,\(^6\) we know that its general solution may be presented as

\[ \chi(r, t) = \chi_{\text{out}} \left( t - \frac{r}{v} \right) + \chi_{\text{in}} \left( t + \frac{r}{v} \right), \] (8.9)

where \( \chi_{\text{in}} \) and \( \chi_{\text{out}} \) are (so far) arbitrary functions of one variable. The physical sense of \( \phi_{\text{out}} = \chi_{\text{out}}/r \) is a spherical wave propagating from our source (at \( r = 0 \)) to outer space, i.e. exactly the solution we are looking for. On the other hand, \( \phi_{\text{in}} = \chi_{\text{in}}/r \) describes a spherical wave that could be created by some distant spherically-symmetric source, that converges on our charge located at the origin – evidently not the effect we want to consider here. Discarding this term, and returning to \( \phi = \chi/r \), we can write the solution (7) as

\[ \phi(r, t) = \frac{1}{r} \chi_{\text{out}} \left( t - \frac{r}{v} \right). \] (8.10)
In order to find function $\chi_{\text{out}}$, let us consider distances $r$ so small that the time derivative in Eq. (3a), with the right-hand part (5),

$$
\nabla^2 \phi - \frac{1}{v^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{q(t)}{\varepsilon}\delta(\mathbf{r}),
$$

(8.11)
is much smaller that the spatial derivative (that diverges at $r \to 0$). Then Eq. (11) is reduced to the electrostatic equation whose solution (4a), for source (5), is

$$
\phi(r \to 0, t) = \frac{q(t)}{4\pi \varepsilon r}.
$$

(8.12)

Now requiring the two solutions, (10) and (12), to coincide at $r \ll vt$, we get $\chi_{\text{out}}(t) = q(t)/4\pi \varepsilon r$, so that Eq. (10) becomes

$$
\phi(r, t) = \frac{1}{4\pi \varepsilon r} q\left( t - \frac{r}{v} \right).
$$

(8.13)

Just as had been done in statics, this result may be readily generalized for the arbitrary position $\mathbf{r}'$ of the point charge:

$$
\rho(\mathbf{r}, t) = q(t)\delta(\mathbf{r} - \mathbf{r}') \equiv q(t)\delta(\mathbf{R}),
$$

(8.14)

where $R$ is the distance between the field observation point $\mathbf{r}$ and the source position point $\mathbf{r}'$, i.e. the length of the vector,

$$
\mathbf{R} = \mathbf{r} - \mathbf{r}',
$$

(8.15)

connecting these points - see Fig. 1.

![Fig. 8.1. Calculating retarded potentials of a localized system.](image)

Obviously, Eq. (13) becomes

$$
\phi(\mathbf{r}, t) = \frac{1}{4\pi \varepsilon R} q\left( t - \frac{R}{v} \right).
$$

(8.16)

Now we can use the linear superposition principle to write, for the arbitrary charge distribution $\rho(\mathbf{r}, t)$,

$$
\phi(\mathbf{r}, t) = \frac{1}{4\pi \varepsilon} \int \rho\left( \mathbf{r}', t - \frac{R}{v} \right) \frac{d^3 r'}{R},
$$

(8.17a)

where integration is extended over all charges of the system under analysis. Acting absolutely similarly, for the vector potential we get
Solutions (17) are called the *retarded potentials*, the name signifying that the observed fields are “retarded” (delayed) in time by \( \Delta t = R/v \) relative to the source variations, due to the finite speed \( v \) of the electromagnetic wave propagation. These solutions are so important that they deserve at least a couple of general remarks.

First, remarkably, these simple expressions are *exact* solutions of the Maxwell equations (93) in a uniform medium for an arbitrary distribution of stand-alone charges and currents. They also may be considered as the *general* solutions of these equations, provided that the integration is extended over all field sources in the Universe – or at least in its part that affects our observations.

Second, if functions \( \rho(r, t) \) and \( j(r, t) \) include the microscopic (bound) charges and currents as well, the macroscopic Maxwell equations (6.93) are valid with the replacement \( \varepsilon \rightarrow \varepsilon_0 \) and \( \mu \rightarrow \mu_0 \), so that the retarded potentials solutions (17) are also valid - with the same replacement.

Finally, Eqs. (17) may be plugged into Eqs. (1), giving (after an explicit differentiation) the so-called *Jefimenko equations* for fields \( E \) and \( B \) – similar in structure to Eqs. (17), but more cumbersome. Conceptually, the existence of such equations is a good news, because they are free from the gauge ambiguity pertinent to potentials \( \phi \) and \( A \). However, the practical value of these explicit expressions for the fields is not too high: for all applications I am aware of, it is easier to use Eqs. (17) to calculate the particular expressions for the potentials first, and only then calculate the fields from Eqs. (1). Let me present the (apparently most important) example of this approach.

### 8.2. Electric dipole radiation

Consider again the problem that was discussed in electrostatics (Sec. 3.1), namely the field of a localized source with linear dimensions \( a \ll r \) (Fig. 1), but now with time-dependent charge and/or current distribution. Using the arguments of that discussion, in particular the condition expressed by Eq. (3.1), \( r' \ll r \), we may apply the Taylor expansion (3.3),

\[
\left. f(R) = f(r) - r' \cdot \nabla f(r) + \ldots \right. 
\]

(8.18)

to function \( f(R) \equiv R \) (for which \( \nabla f(r) = \nabla R = n \), where \( n \equiv r/r \) is the unit vector directed toward the observation point, see Fig. 1) to approximate distance \( R \) as

\[
R \approx r - r' \cdot n. 
\]

(8.19)

In each of the retarded potential formulas (17), \( R \) participates in two places: in the denominator and in the source time argument. If \( \rho \) and \( j \) change in time on scale \( \sim 1/\omega \), where \( \omega \) is some characteristic frequency, then any change of argument \( (t - R/v) \) on that time scale, for example due to a change of \( R \) on the spatial scale \( \sim v/\omega = 1/k \), may substantially change these functions. Thus, expansion (18) may be applied to \( R \) in the argument \( (t - R/v) \) only if \( ka \ll 1 \), i.e. if the system size \( a \) is much smaller than the radiation wavelength \( \lambda = 2\pi/k \). On the other hand, function \( 1/R \) changes relatively slowly, and for it even the first term expansion (19) gives a good approximation as soon as \( a \ll r, R \). In this approach, Eq. (17a) yields
\[ \phi(r,t) \approx \frac{1}{4\pi r} \int \rho \left( r', t - \frac{R}{v} \right) d^3r' = \frac{1}{4\pi r} Q \left( t - \frac{R}{v} \right), \quad (8.20) \]

where \( Q(t) \) is the net electric charge of the localized system. Due to the charge conservation, this charge cannot change with time, so that the approximation (20) describes gives just a static Coulomb field of our localized source, rather than a radiated wave.

Let us, however, apply a similar approximation to the vector potential (17b):

\[ A(r,t) \approx \frac{\mu}{4\pi r} \int j \left( r', t - \frac{R}{v} \right) d^3r'. \quad (8.21) \]

According to Eq. (5.87), in statics the right-hand part of this expression would vanish, but in dynamics this is no longer true. For example, if the current is due to a non-relativistic motion\(^7\) of a system of charges \( q_k \), we can write

\[ \int j(r', t) d^3r' = \sum_k q_k \dot{r}_k(t) = \frac{d}{dt} \sum_k q_k r_k(t) \equiv \dot{p}(t), \quad (8.22) \]

where \( \dot{p}(t) \) is the dipole moment of the localized system, defined by Eq. (3.6). Now, after the integration, we may keep only the first term of approximation (19) in the argument \((t - R/v)\) as well, getting

\[ A(r,t) \approx \frac{\mu}{4\pi r} \dot{p} \left( t - \frac{r}{v} \right). \quad (8.23) \]

Let us analyze what exactly does this result, valid in the limit \( ka << 1 \), describe. The second of Eqs. (1) allows us to calculate the magnetic field by the spatial differentiation of \( A \). At large distances \( r >> \lambda \) (i.e. in the so-called far field zone), where Eq. (23) describes a virtually plane wave, the main contribution into this derivative is given by the dipole moment factor:

\[ \mathbf{B}(r,t) = -\frac{\mu}{4\pi r} \nabla \times \dot{p} \left( t - \frac{r}{v} \right) = -\frac{\mu}{4\pi rv} \mathbf{n} \times \dot{p} \left( t - \frac{r}{v} \right). \quad (8.24) \]

This expression means that the magnetic field, at the observation point, is perpendicular to vectors \( \mathbf{n} \) and \((\text{the retarded value of})\ \dot{p} \), and its magnitude is

\[ B = \frac{\mu}{4\pi rv} \dot{p} \left( t - \frac{r}{v} \right) \sin \theta, \quad \text{i.e.} \quad H = \frac{1}{4\pi rv} \dot{p} \left( t - \frac{r}{v} \right) \sin \theta, \quad (8.25) \]

where \( \theta \) is the angle between those two vectors – see Fig. 2.\(^8\)

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\(^7\) For relativistic particles, moving with velocities of the order of speed of light, one has to be more careful. As the result, I will postpone the discussion of their radiation until Chapter 10, i.e. until after the discussion of special relativity in Chapter 9.

\(^8\) From the first of Eqs. (1), for the electric field, in the first approximation (23), we would get \(-\partial A/\partial t = -(1/4\pi rv)\ \dot{p} \left( t - r/v \right) = -(Z/4\pi) \mathbf{p} \left( t - r/v \right). \) The transverse component of this vector (see Fig. 2) is the proper wave field \( \mathbf{E} = Z\mathbf{H} \times \mathbf{n} \), while its longitudinal component is exactly compensated by \((\nabla \phi)\) in the next term of expansion of Eq. (17a) with respect to small parameter \( r/\lambda << 1 \).
The most important feature of this result is that the time-dependent field decreases very slowly (only as $1/r$) with the distance from the source, so that the radial component of the corresponding Poynting vector (7.7), $S_r = ZH^2$, drops as $1/r^2$, i.e. the full power $P$ of the emitted spherical wave, that scales as $r^2S_r$, does not depend on the distance from the source – as it should for radiation. Equation (25) allows us to be more quantitative; for the instantaneous radiation intensity we may plug it into Eq. (7.9) to get

$$S_r = ZH^2 = \frac{Z}{(4\pi vr)^2} \left( \ddot{p} \left( t - \frac{r}{v} \right) \right)^2 \sin^2 \theta. \quad (8.26)$$

This is the famous formula for the electric dipole radiation; this is the dominating component of radiation by a localized system of charges - unless $\ddot{p} = 0$. Please notice its angular dependence: the radiation vanishes at the axis of the retarded vector $\ddot{p}$ (where $\theta = 0$), and reaches its maximum in the plane perpendicular to that axis. Integration of $S_r$ over all directions, i.e. over the whole sphere of radius $r$, gives the total instant power of the dipole radiation:

$$P \equiv \oint_{r=\text{const}} S_r \, d^2 r = \frac{Z}{(4\pi v)^2} \ddot{p}^2 \frac{r^3}{2\pi} \int_0^{\pi} \sin^3 \theta \, d\theta = \frac{Z}{6\pi v^2} \ddot{p}^2. \quad (8.27)$$

In order to find the average power, this expression has to be averaged over a sufficiently long time. In particular, if the source is monochromatic, $p(t) = \Re[p_\omega \exp\{-i\omega t\}]$, with time-independent vector $p_\omega$, such averaging may be carried out just over one period, giving an extra factor 2 in the denominator:

$$\overline{P} = \frac{Z\omega^3}{12\pi v^2} |p_\omega|^2. \quad (8.28)$$

The easiest example of application of the formula is to a point charge oscillating, with frequency $\omega$, along a straight line (that we may take for axis $z$), with amplitude $a$. In this case, $p = qn_\omega z(t) = qa \Re \{\exp\{-i\omega t\}\}$, and if the charge velocity amplitude, $a\omega$, is much less than the wave speed $v$, we may use Eq. (28) with $p_\omega = qa$, giving

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9 In the Gaussian units, for free space ($v = c$), this important formula reads $P = (2/3c^3)\ddot{p}^2$. It was first derived in 1897 by J. Larmor for the particular case of a single point charge $q$ moving with acceleration $\ddot{r}$, when $\ddot{p} = q\ddot{r}$ and hence $P = (2q^2/3c^3)\ddot{r}^2$. As a result, Eq. (27) is sometimes referred to as the Larmor formula.
Applied to an electron \( (q = -e \approx -1.6 \times 10^{-19} \text{ C}) \), rotating about a nuclei at an atomic distance \( a \sim 10^{-10} \text{ m} \), the Larmor formula shows\(^{10}\) that the energy loss due to the dipole radiation is so large that it would cause electron’s collapse on atom’s nuclei in just \( \sim 10^{-10} \text{ s} \). In the beginning of the 1900s, this classical result was one of the main arguments for the development of quantum mechanics that prevents such collapse of electrons in their lowest-energy (ground) state.

Another example of a very useful application of Eq. (28) is the radio wave radiation by a short, straight, symmetric antenna which is fed, for example, by a TEM transmission line such as a coaxial cable – see Fig. 3.

The exact solution of this problem is rather complex, because the law \( I_\omega(z) \) of the current variation along antenna’s length should be calculated self-consistently with the distribution of the electromagnetic field that is induced by the current in the surrounding space. (This fact is unfortunately ignored in some textbooks.) However, one may argue that at \( l \ll \lambda \), the current should be largest in the feeding point (in Fig. 3, taken for \( z = 0 \)), vanish at antenna’s ends \( (z = \pm l/2) \), and that the only possible scale of the current variation in the antenna is \( l \) itself, so that the linear function,

\[
I_\omega(z) = I_\omega(0) \left( 1 - \frac{2}{l} |z| \right),
\]

(8.30)
gives a good approximation - as it indeed does. Now we can use the continuity equation \( \frac{\partial Q}{\partial t} = I \), i.e. \( -i\omega Q_\omega = I_\omega \), to calculate the complex amplitude \( Q_\omega(z) = iI_\omega(z) \text{sgn}(z) / \omega \) of the electric charge \( Q(z, t) = \text{Re}[Q_\omega \exp{-i\omega t}] \) of the wire beyond point \( z \), and from it, the amplitude of the linear density of charge

\[
\lambda_\omega(z) = \frac{d Q_\omega(z)}{dz} = -i \frac{2I_\omega(0)}{\omega l} \text{sgn} z.
\]

(8.31)

From here, the dipole moment’s amplitude is

\[
p_\omega = 2 \int_{-l/2}^{l/2} \lambda_\omega(z) zdz = -i \frac{I_\omega(0)}{2\omega} l,
\]

(8.32)

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\(^{10}\) Actually, the formula needs a numerical coefficient adjustment to account for electron’s orbital (rather than linear) motion – the task left for reader’s exercise. However, this adjustment does not affect the order-of-magnitude estimate given above.
so that Eq. (28) yields

$$\overline{\mathcal{P}} = Z \frac{\omega^4}{12\pi v^2} \frac{|I_\omega(0)|^2}{4\omega^2} l^2 = \frac{Z(kl)^2}{24\pi} \frac{|I_\omega(0)|^2}{2},$$  \hspace{1cm} (8.33)

where \( k = \omega v \). The analogy between this result and the dissipation power, \( \mathcal{P} = \text{Re} \{ I_\omega^2 / 2 \} \), in a lumped linear circuit element, allows the interpretation of the first fraction in the last form of Eq. (33) as the real part of antenna’s impedance:

$$\text{Re} Z_A = Z \frac{(kl)^2}{24\pi}, \hspace{1cm} (8.34)$$

as felt by the transmission line. (Indeed, according to Eq. (7.118), the wave traveling along the line toward the antenna is fully radiated, i.e. not reflected back, only if \( Z_A \) equals to \( Z_W \) of the line.) As we know from Chapter 7, for typical TEM lines, \( Z_W \sim Z_0 \), while Eq. (34), that is only valid in the limit \( kl \ll 1 \), shows that for radiation into free space (\( Z = Z_0 \)), \( \text{Re} Z_A \) is much less than \( Z_0 \).

Hence in order to reach the impedance matching condition \( Z_W = Z_A \), antenna’s length should be increased – as a more involved theory shows, to \( l \sim \lambda/2 \). However, in many cases, practical considerations make short antennas necessary. The most frequently met example met nowadays are the cell phone antennas, which use frequencies close to 1 or 2 GHz, with free-space wavelengths \( \lambda \) between 15 and 30 cm, i.e. much larger than the phone size. The quadratic dependence of antenna’s efficiency on \( l \), following from Eq. (34), explains why every millimeter counts in the design of such antennas, and why the designs are carefully optimized using software packages for (virtually exact) numerical solution of time-dependent Maxwell equations for the specific shape of the antenna and other phone parts.\(^{11}\)

To conclude this section, let me note that if the wave source is not monochromatic, so that \( p(t) \) should presented as a Fourier series,

$$p(t) = \text{Re} \sum_\omega p_\omega e^{-i\omega t}, \hspace{1cm} (8.35)$$

the terms corresponding to interference of spectral components with different frequencies \( \omega \) are averaged out at the time averaging of the Poynting vector, so that the average radiated power is just a sum of contributions (28) from all substantial frequency components.

### 8.3. Wave scattering

The formalism described above may be immediately used in the theory of scattering – the phenomenon illustrated by Fig. 4. Generally, scattering is a complex problem. However, in many cases it allows the so-called Born approximation,\(^{12}\) in which scattered wave’s field applied to the scattering object is assumed to be much weaker than that of the incident wave, and is neglected.

\( ^{11} \) A partial list of popular software packages of this kind includes both publicly available codes such as NEC -2 (whose various versions are available online, e.g., at http://alioth.debian.org/projects/necpp/ and http://www.qsl.net/4nec2/), and proprietary packages - such as Momentum from Aglient Technologies (now owned by Hewlett-Packard), FEKO from EM Software & Systems, and XFdit from Remcom.

\( ^{12} \) Named after M. Born, one of the founding fathers of quantum mechanics. Note, however, the basic idea of this approach was developed much earlier (in 1881) by Lord Rayleigh – see below.
As the first example of this approach, let us consider scattering of a plane wave, propagating in free space \((Z = Z_0, v = c)\), by a free\(^{13}\) charged particle whose motion may be described by non-relativistic classical mechanics. (This requires, in particular, the incident wave to be of a modest intensity, so that the speed of the induced charge motion is much less than the speed of light.) In this case the magnetic component of the Lorentz force (5.8),

\[
\mathbf{F}_m = q\mathbf{r} \times \mathbf{B},
\]

exerted on the charge by the magnetic field of a plane wave, is much smaller than force \(\mathbf{F}_e = q\mathbf{E}\) exerted by its electric field. Indeed, according to Eq. (7.8), \(H = E/Z = E/(\mu_0\epsilon) = \mu H = E/v\), so that the ratio \(F_m/F_e\) equals to the ratio of particle’s speed, \(|\mathbf{r}|\), to wave’s speed \(v \sim c\).

Thus, assuming that the incident wave is linearly-polarized along axis \(x\), the equation of particle’s motion in the Born approximation is just \(m\ddot{x} = q\mathbf{E}(t)\), so that for the \(x\)-component \(p_x = qx\) of its dipole moment we can write

\[
\ddot{x} = \frac{q^2}{m} E(t).
\]

As we already know from Sec. 2, oscillations of the dipole moment lead to radiation of a wave with a wide angular distribution of intensity; in our case this is the scattered wave – see Fig. 4. Its full power may be found by plugging Eq. (37) into Eq. (27):

\[
\mathcal{P} = \frac{Z_0}{6\pi c^2} \bar{p}^2 = \frac{Z_0 q^4}{6\pi c^2 m^2 E^2(t)}, \text{ i.e. } \bar{\mathcal{P}} = \frac{Z_0 q^4}{12\pi c^2 m^2 |E_\omega|^2}.
\]

Since the power is proportional to incident wave’s intensity \(S\), it is customary to characterize scattering ability of the object by the ratio,

\[
\sigma = \frac{\bar{\mathcal{P}}}{S_{\text{incident}}} = \frac{\bar{\mathcal{P}}}{|E_\omega|^2 / 2Z_0},
\]

which evidently has the dimension of area and is called the full cross-section of scattering. For this measure, Eq. (38) yields the famous result

\[\sigma \approx \frac{Z_0}{12\pi c^2 m^2 |E_\omega|^2}.
\]

\(^{13}\) As Eq. (7.30) shows, this calculation is also valid for an oscillator with eigenfrequency \(\omega_b \ll \omega\).
\[ \sigma = \frac{Z^2 q^4}{6\pi e^2 m^2} = \frac{\mu_0^2 q^4}{6\pi m^2}, \]  
(8.40)

which is called the Thomson scattering formula,\(^{14}\) especially when applied to an electron. This relation is most frequently presented in the form\(^{15}\)

\[ \sigma = \frac{8\pi}{3} r_c^2, \quad \text{with } r_c \equiv \frac{q^2}{4\pi\varepsilon_0 mc} = 10^{-7} \frac{q^2}{m}. \]  
(8.41)

Constant \(r_c\) is called the **classical radius of the particle** (or sometimes the “Thomson scattering length”); for electron (\(q = -e, m = m_e\)) it is close to \(2.82 \times 10^{-15}\) m. Its possible interpretation is evident from the first form of Eq. (41) for \(r_c\): at that distance between two similar particles, the potential energy \(q^2/4\pi\varepsilon_0 r\) of their electrostatic interaction is equal to particle’s rest-mass energy \(mc^2\).\(^{16}\)

Now we have to go back and establish the conditions at which the Born approximation, when the field of the scattered wave is negligible, is indeed valid for a point-object scattering. Since the scattered wave’s intensity, described by Eq. (26), diverges as \(1/r^2\), according to the definition (39) of the cross-section, it may become comparable to \(S_{\text{incident}}\) at \(r^2 \sim \sigma\). However, Eq. (38) itself is only valid if \(r >> \lambda\), so that the Born approximation does not lead to any contradiction if

\[ \sigma << \lambda^2. \]  
(8.42)

For the Thompson scattering by an electron, this condition means \(\lambda >> r_c \sim 3 \times 10^{-15}\) m and is fulfilled for all frequencies up to very hard \(\gamma\) rays with energies \(\sim 100\) MeV.

Possibly the most notable feature of result (40) is its independence of the wave frequency. As it follows from its derivation, particularly from Eq. (37), this independence is intimately related with the unbound character of charge motion. For bound charges, say for electrons in a gas molecule, this result is only valid if the wave frequency \(\omega\) is much higher all eigenfrequencies \(\omega_j\) of molecular resonances. In the opposite limit, \(\omega << \omega_j\), the result is dramatically different. Indeed, in this limit we can approximate the molecule’s dipole moment by its static value (3.39)

\[ \mathbf{p} = 4\pi\varepsilon_0 \alpha \text{mol} \mathbf{E}. \]  
(8.43)

In the Born approximation, and in the absence of the molecular field effects discussed in Sec. 3.5, \(\mathbf{E}\) in this expression is just the incident wave’s field, and we can use Eq. (28) to calculate the power of the wave scattered by a single molecule:

---

\(^{14}\) Named after Sir J. J. Thomson (1856-1940), the discoverer of the electron - and isotopes as well! He is not to be confused with his son, G. P. Thomson, who discovered (simultaneously with C. Davisson and L. Germer) quantum-mechanical wave properties of the same electron.

\(^{15}\) In the Gaussian units, this formula looks like \(r_c = q^2/mc^2\) (giving, of course, the same numerical value: for the electron, \(r_c \approx 2.82 \times 10^{-13}\) cm). This **classical** quantity should not be confused with particle’s **Compton wavelength** \(\lambda_c = \hbar/mc\) (for the electron, close to \(2.24 \times 10^{-12}\) m), which naturally arises in **quantum** electrodynamics – see a brief discussion in the next chapter, and QM Chapter 9 for more detail.

\(^{16}\) It is fascinating how smartly has the **relativistic** expression \(mc^2\) sneaked into the result (40), which was obtained using a **non-relativistic** equation of particle motion. This was possible because the calculation engaged electromagnetic waves that propagate with the speed of light, and whose quanta (**photons**), as a result, may be frequently treated as relativistic (moreover, ultra-relativistic) particles - see the next chapter.
Now, using the last form of definition (39) of the cross-section, we get a very simple result,

\[ \sigma = \frac{8\pi Z_0^2 \varepsilon_0^2 \omega^4}{3e^2} \alpha_{\text{mol}}^2 = \frac{8\pi k^4}{3} \alpha_{\text{mol}}^2, \]  

(8.45)

showing that in contrast to Eq. (40), at low frequencies \( \sigma \) grows as fast as \( \omega^4 \).

Now let us explore the effect of such Rayleigh scattering\(^{17} \) on wave propagation in a gas, with relatively low density \( n \). We can expect (and will prove in the next section) that due to the randomness of molecule positions, the waves scattered by each molecules may be treated as incoherent, so that the total scattering power may be calculated just as the sum of those scattered by each molecule. We can use this additivity to write the balance of the incident’s wave intensity on a small volume \( dV \) of length (along the incident wave direction) \( dz \), and area \( A \) in across it. Since such a segment includes \( n dV = nAdz \) molecules, and, according to definition (39), each of them scatters power \( S\sigma = \mathcal{P}\sigma/A \), the total scattered power is \( n\mathcal{P}\sigma dz \); hence the incident power’s change is

\[ d\mathcal{P} = -n\sigma \mathcal{P} dz. \]  

(8.46)

Comparing this equation with the general definition (7.202) of the attenuation constant, we see that scattering gives the following contribution to attenuation: \( \alpha = n\sigma \). From here, using Eq. (3.41) to write \( \alpha_{\text{mol}} = (\varepsilon_r - 1)/4\pi n \), and Eq. (45), we get

\[ \alpha = \frac{k^4}{6\pi n} (\varepsilon_r - 1)^2. \]  

(8.47)

This is the famous Rayleigh scattering formula, which in particular explains the colors of blue sky and red sunsets. Indeed, through the visible light spectrum, \( \omega \) changes almost two-fold; as a result, scattering of blue components of sunlight is an order of magnitude higher than that of its red components. More qualitatively, for air near the Earth surface, \( \varepsilon_r - 1 \approx 6 \times 10^4 \), and \( n \approx 2.5 \times 10^{-5} \) m\(^{-3} \) - see Sec. 3.3. Plugging these numbers into Eq. (47), we see that the characteristic length \( L \equiv 1/\alpha \) of scattering is \( \approx 30 \) km for blue light and \( \approx 200 \) km for red light.\(^{18} \) The Earth atmosphere is thinner (\( h \approx 10 \) km), so that the Sun looks just a bit yellowish during most of the day. However, elementary geometry shows that on sunset, the light should pass length \( l \approx (R_E h)^{1/2} \approx 300 \) km to reach an Earth-surface observer; as a result, the blue components of Sun’s light spectrum are almost completely scattered out, and even the red components are weakened considerably.

To conclude the discussion of Eq. (47), let me note that its comparison with the condition of the direct applicability of the Born approximation for a distributed object of size \( a \):

\[ \alpha a \ll 1, \]  

(8.48)

\(^{17}\) Named after Lord Rayleigh (born J. Strutt, 1842-1919), whose numerous contributions to science include the discovery of argon. He has also pioneered (for the special case we are considering now) the basic idea of what is presently called the Born approximation.

\(^{18}\) These values are approximate because both \( n \) and \( (\varepsilon_r - 1) \) vary through the atmosphere.
implies, in particular, that if the electric polarizability of the material is small, \( \varepsilon_r \to 1 \), we may be able to use the approximation for an analysis of scattering by even relatively large objects, with size of the order of, or even larger than \( \lambda \). However, for such extended objects, the phase difference factors (neglected above) step in, leading in particular to the important effects of interference and diffraction, to whose discussion we now proceed.

### 8.4. Interference and diffraction

These effects show up not as much in the total power of scattered radiation, as in its angular distribution. It is traditional to characterize this distribution by the differential cross-section defined as

\[
\frac{d\sigma}{d\Omega} = \frac{S_r r^2}{S_{\text{incident}}}, \quad (8.49)
\]

where \( r \) is the distance from the scatterer, at which the scattered wave is observed. Both the definition and notation become more clear if we notice that according to Eq. (26), at large distances \( (r >> a) \), the numerator in the right-hand part of Eq. (49), and hence the differential cross-section as the whole, does not depend on \( r \), and that its integral over the total solid angle \( \Omega = 4\pi \) coincides with the total cross-section defined by Eq. (39):

\[
\int_4^\pi \frac{d\sigma}{d\Omega} d\Omega = \frac{1}{S_{\text{incident}}} \int_4^\pi r^2 S_r d\Omega = \frac{1}{S_{\text{incident}}} \int_4^\pi S_r d^2r = \frac{\bar{\rho}}{S_{\text{incident}}} \equiv \sigma . \quad (8.50)
\]

For example, according to Eq. (26), the angular distribution of radiation scattered by a point linear dipole, in the Born approximation, is rather broad; in particular, in the low-frequency limit (43),

\[
\frac{d\sigma}{d\Omega} = k^4 \alpha_{\text{mol}}^2 \sin^2 \theta . \quad (8.51)
\]

If the wave is scattered by a small dielectric body, with a characteristic size \( a << \lambda \) (i.e., \( ka << 1 \)), then all its parts re-radiate the incident wave coherently. Hence, we can calculate it in the similar way, just replacing the molecular dipole moment (43) with the total dipole moment of the object – see Eq. (3.37):

\[
\mathbf{p} = PV = (\varepsilon_r - 1)\varepsilon_0 \mathbf{E} V, \quad (8.52)
\]

where \( V \sim a^3 \) is body’s volume. As a result, the differential cross-section may be obtained from Eq. (51) with the replacement \( \alpha_{\text{mol}} \to V(\varepsilon_r - 1)/4\pi \):

\[
\frac{d\sigma}{d\Omega} = k^4 V^2 (\varepsilon_r - 1)^2 \sin^2 \theta , \quad (8.53)
\]

i.e. follows the same \( \sin^2 \theta \) law. The situation for extended objects, with at least one dimension of the order, or larger than the wavelength, is different: here we have to take into account that the phase shifts introduced by various parts of the body are different. Let us analyze this issue for an arbitrary collection of similar point scatterers located at points \( \mathbf{r}_j \).

If wave vector of the incident plane wave is \( \mathbf{k}_0 \), the field the wave has the phase factor \( \exp\{i\mathbf{k}_0 \cdot \mathbf{r}_j\} \) – see Eq. (7.79). At the location of \( j \)-th scattering center, the factor equals to \( \exp\{i\mathbf{k}_0 \cdot \mathbf{r}_j\} \), so that the local polarization vector \( \mathbf{p}_j \) and the scattered wave it creates, are proportional to this factor. On
its way to the observation point \( \mathbf{r} \), the scattered wave, with wave vector \( \mathbf{k} \) (with \( k = k_0 \)), acquires an additional phase factor \( \exp \{ ik \mathbf{r} \cdot (\mathbf{r} - \mathbf{r}_j) \} \), so that the scattered wave field is proportional to

\[
\exp \{ ik_0 \cdot \mathbf{r}_j + ik \mathbf{r} \cdot (\mathbf{r} - \mathbf{r}_j) \} = e^{ik \cdot r} \exp \{ -i(k - k_0 \cdot \mathbf{r}_j) \}. \tag{8.54}
\]

Since the first factor in the last expression does not depend on \( \mathbf{r}_j \), in order to calculate the total scattering wave, it is sufficient to sum up the elementary phase factors \( \exp \{-i \mathbf{q} \cdot \mathbf{r}_j \} \), where vector

\[
\mathbf{q} \equiv \mathbf{k} - k_0 \tag{8.55}
\]

has the physical sense of the wave vector change at scattering.\(^{19}\) It may look like the phase factor depends on the choice of origin. However, according to Eq. (7.42), the average intensity of the scattered wave is proportional to \( E_\omega E_\omega^* \), i.e. to the following real scalar function of vector \( \mathbf{q} \):

\[
F(\mathbf{q}) = \left( \sum_j \exp \{-i \mathbf{q} \cdot \mathbf{r}_j \} \right) \left( \sum_j \exp \{-i \mathbf{q} \cdot \mathbf{r}_j \}^* \right) = \sum_{j,j'} \exp \{i \mathbf{q} \cdot (\mathbf{r}_j - \mathbf{r}_{j'}) \} \equiv |I(\mathbf{q})|^2, \tag{8.56}
\]

where the complex function

\[
I(\mathbf{q}) \equiv \sum_j \exp \{-i \mathbf{q} \cdot \mathbf{r}_j \} \tag{8.57}
\]

is called the phase sum, may be calculated within any reference frame, without affecting the final result (56). The double-sum form of Eq. (56) is convenient to notice that for a system of many \((N >> 1)\) of similar but randomly located scatterers, only the terms with \( j = j' \) accumulate at summation, so that \( F(\mathbf{q}) \) scales as \( N \), rather than \( N^2 \) - thus justifying the above treatment of the Rayleigh scattering problem.

Let us start using Eq. (56) by applying it to the simplest problem of just two similar small scatterers, separated by a fixed distance \( a \):

\[
F(\mathbf{q}) = \sum_{j,j'=1}^2 \exp \{i \mathbf{q} \cdot (\mathbf{r}_j - \mathbf{r}_{j'}) \} = 2 + \exp \{-i \mathbf{q}_a \cdot a \} + \exp \{i \mathbf{q}_a \cdot a \} = 2(1 + \cos q_a \cdot a) = 4 \cos^2 \frac{q_a \cdot a}{2}, \tag{8.58}
\]

where \( q_a \equiv \mathbf{q} \cdot \mathbf{a} / a \) is the component of vector \( \mathbf{q} \) along vector \( \mathbf{a} \) connecting the scatterers. The apparent simplicity of this result may be a bit misleading, because the mutual plane of vectors \( \mathbf{k} \) and \( k_0 \) (and hence of vector \( \mathbf{q} \)) does not necessarily coincide with the mutual plane of vectors \( k_0 \) and \( E_\omega \), so that the scattering angle \( \alpha \) between vectors \( \mathbf{k} \) and \( k_0 \) is generally different from \((\pi/2 - \theta)\) - see Fig. 5.

\[\text{Fig. 8.5. Angles important for the general scattering problem.}\]

\(^{19}\) In quantum electrodynamics, \( \hbar \mathbf{q} \) has the sense of the momentum transferred from the scattering object to the scattered photon, and this terminology sometimes creeps even into the classical electrodynamic texts.
Moreover, vectors \( \mathbf{q} \) and \( \mathbf{a} \) may have another common plane, and angle between them is one more parameter that may be considered as independent from both \( \alpha \) and \( \theta \). As a result, the angular dependence of the scattered wave’s intensity (and hence \( d\sigma/d\Omega \)), that depends on all three angles, may be rather complex.

This is why let me consider only the simple case when vectors \( \mathbf{k}, \mathbf{k}_0, \) and \( \mathbf{a} \) are all in the same plane (Fig. 6a), with \( \mathbf{k}_0 \) perpendicular to \( \mathbf{a} \) (leaving the general analysis for readers’ exercise). Then, with our choice of coordinates, \( q_a = q_x = k \sin \alpha \), and Eq. (58) is reduced to

\[
F(q) = 4 \cos^2 \frac{ka \sin \alpha}{2}.
\]

This function always has two maxima, at \( \alpha = 0 \) and \( \alpha = \pi \), and possibly (if the product \( ka \) is large enough) other maxima at special angles \( \alpha_n \) that satisfy the famous Bragg condition\(^{20}\)

\[
ka \sin \alpha_n = 2\pi n, \quad \text{i.e. } a \sin \alpha_n = n\lambda. \tag{8.60}
\]

As evident from Fig. 6a, this condition may be readily understood as the in-phase addition (frequently called the constructive interference) of two coherent waves scattered by the two points, when the difference between their paths toward the observer, \( a \sin \alpha \), equals to an integer number of wavelengths. At each such maximum, \( F = 4 \), due to the doubling of the wave amplitude and hence quadrupling its power.

If the distance between the point scatterers is large (\( ka >> 1 \)), the first Bragg maxima correspond to small angles, \( \alpha << 1 \). For this region, Eq. (59) in reduced to a simple sinusoidal dependence of function \( F \) on angle \( \alpha \). Moreover, within the range of small \( \alpha \), the polarization factor \( \sin^2 \theta \) is virtually constant, so that the scattered wave intensity, and hence the differential cross-section

\[
\frac{d\sigma}{d\Omega} \propto F(q) = 4 \cos^2 \frac{ka \alpha}{2}. \tag{8.61}
\]

This is of course the well-known interference pattern, well known from the Young’s two-slit experiment.\(^{21}\) (As will be discussed in the next section, theoretical description of the two-slit experiment

---

\(^{20}\) Named after Sir William Bragg and his son, Sir William Lawrence Bragg, who in 1912 demonstrated X-ray diffraction by atoms in crystals. The Braggs’ experiments have made the existence of atoms (before that, a hypothetical notion ignored by many physicists) indisputable.
is more complex than that of the Born scattering, but is preferable experimentally, because at scattering, the wave of intensity (61) has to be observed on the backdrop of a stronger incident wave that propagates in almost the same direction, $\alpha = 0$.)

The Bragg condition (60) does not change at scattering from $N > 2$ similar, equidistant scatterers, located along the same straight line (because the condition is applicable to each pair of adjacent scatterers), but the interference pattern changes. Leaving the analysis of the case of arbitrary $N$ for reader’s exercise, let me jump to the limit $N \to 0$, in which we may ignore the scatterer discreteness. The resulting pattern is similar to that at scattering by a continuous thin rod, so let us first discuss the Born scattering by an arbitrary distributed object - say an extended dielectric body with a constant value of $\varepsilon_r$. Transferring Eq. (56) from the sum to an integral, for the differential cross-section we get

$$
\frac{d\sigma}{d\Omega} = k^4 (4\pi)^2 (\varepsilon_r - 1)^2 F(\mathbf{q}) \sin^2 \theta = -\frac{k^4}{(4\pi)^2} (\varepsilon_r - 1)^2 |I(\mathbf{q})|^2 \sin^2 \theta ,
$$

where $I(\mathbf{q})$ now becomes the phase integral,$^{22}$

$$
I(\mathbf{q}) = \int_V \exp\{-i\mathbf{q} \cdot \mathbf{r}'\} d^3r',
$$

with the dimensionality of volume.

Now we may return to the particular case of a thin rod (with both dimensions of the cross-section’s area much smaller than $\lambda$, but an arbitrary length $a$), otherwise keeping the same simple geometry as for two point scatterers – see Fig. 6b. In this case the phase integral is just

$$
I(\mathbf{q}) = A \int_{-a/2}^{+a/2} \exp\{-iq_x x'\} dx' = A \frac{\exp\{-iq_x a/2\} - \exp\{-iq_x a/2\}}{-iq} = V \frac{\sin \xi}{\xi} ,
$$

where $V = Aa$ is the volume of the rod, and $\xi$ is a dimensionless parameter defined as

$$
\xi = \frac{q_x a}{2} = \frac{ka \sin \alpha}{2} .
$$

The fraction participating in Eq. (64) is met in physics so frequently that it has deserved the special name sinc (not “sync”, please!) function:

$$
sinc \xi \equiv \frac{\sin \xi}{\xi} .
$$

Obviously, this function, plotted in Fig. 7, vanishes at all points $\xi = n\pi$, with integer $n$, besides point $n = 0$: sinc $\xi_0 = \text{sinc} 0 = 1$.

---

$^{21}$ This experiment was described as early as in 1803 by T. Young – one more universal genius of science, who has also introduced the Young modulus in the elasticity theory (see, e.g., CM Chapter 7), besides numerous other achievements - including deciphering Egyptian hieroglyphs! The two-slit experiment has firmly established the wave picture of light, to be replaced by the dualistic photon-vs-wave picture, formalized by quantum electrodynamics, only 100+ years later.

$^{22}$ Since the observation point’s position $\mathbf{r}$ does not participate in this formula explicitly, the prime sign in $\mathbf{r}'$ could be dropped, but I keep it as a reminder that the integral is taken over points $\mathbf{r}'$ of the scattering object.
The function $F(q) = V^2 \text{sinc}^2 \xi$, resulting from Eq. (64), is plotted by red line in Fig. 8, and is called the **Fraunhofer diffraction pattern**.

Note that it oscillates with the same argument period $\Delta(\text{ka} \sin \alpha) = \frac{2\pi}{\text{ka}} << 1$ as the interference pattern (59) from two point scatterers (shown with the blue line in Fig. 8). However, at the interference, the scattered wave intensity vanishes at angles $\alpha_n$ that satisfy condition

$$\frac{ka \sin \alpha_n}{2\pi} = n + \frac{1}{2}, \quad (8.67)$$

when the optical paths difference $a \sin \alpha$ equals to a semi-integer number of wavelengths $\lambda/2 = \pi k$, and hence the two waves from the scatterers arrive to the observer in anti-phase (the so-called **destructive interference**). On the other hand, for the diffraction from a continuous rod the minima occur at a different set of angles,

$$\frac{ka \sin \alpha_n}{2\pi} = n, \quad (8.68)$$

i.e. exactly where the two-point interference pattern has its maxima. The reason for this relation is that the wave diffraction on the rod may be considered as a simultaneous interference of waves from all its fragments, and exactly at the observation angles when the rod edges give waves with phases shifted by $2\pi n$, the interior point of the rod give waves with all possible phases, with their algebraic sum equal to
zero. Even more visibly in Fig. 8, at diffraction the intensity oscillations are limited by a rapidly decreasing envelope function $1/\xi^2$. The reason for this fast decrease is that with each Fraunhofer diffraction period, a smaller and smaller fraction of the road gives an unbalanced contribution to the scattered wave.

If rod’s length is small ($ka << 1$, i.e. $a << \lambda$), then sinc’s argument $\xi$ is small at all scattering angles $\alpha$, so $I(q) \approx V$, and Eq. (64) is reduced to Eq. (53). In the opposite limit, $a >> \lambda$, the first zeros of function $I(q)$ correspond to very small angles $\alpha$, for which $\sin \theta \approx 1$, so that the differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{k^4 V^2}{(4\pi)^2} (\varepsilon_r - 1)^2 \sin^2 \frac{ka\alpha}{2}, \quad (8.69)$$

i.e. Fig. 8 shows the scattering intensity as a function of the diffraction direction – if the pattern is observed within the plane containing the rod.

8.5. The Huygens principle

The Born approximation allows tracing the basic features of (and the difference between) the phenomena of interference and diffraction. Unfortunately, this approximation, based on the relative weakness of the scattered wave, cannot be used for more popular experimental implementations of these phenomena, for example, the Young’s two-slit experiment, or diffraction on a single slit or orifice – see, e.g. Fig. 9. Indeed, at such experiments, the orifice size $a$ is typically much larger than light’s wavelength, and as a result, no clear decomposition of the fields to the incident and “scattered” waves is possible.

![Fig. 8.9. Typical geometry for the Huygens principle application.](image)

However, for such experiments, another approximation, called the Huygens (or “Huygens-Fresnel”) principle, is very instrumental: the passed wave may be presented as a linear superposition of spherical waves of the type (17), as if they were emitted by every point of the orifice (or more physically, by every point of the incident wave’s front that has arrived at the orifice). This approximation is valid if the following strong conditions are satisfied:

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23 Named after C. Huygens (1629-1695) who had conjectured the wave theory of light (that remained controversial for more than a century, until T. Young’s experiments), and A.-J. Fresnel (1788-1827) who has developed the mathematical theory of diffraction.
\[ \lambda << a << r , \quad (8.70) \]

where \( r \) is the distance of the observation point from the orifice. In addition, as we have seen in the last section, at small \( \lambda/a \) the diffraction phenomena are confined to angles \( \alpha \sim 1/ka \sim \lambda/a << 1 \). For observation at such small angles, the mathematical expression of the Huygens principle, for a complex amplitude \( f_\omega(r) \) of a monochromatic wave \( f(r, t) = \text{Re}[f_\omega e^{i\omega t}] \), is given by the following simple formula

\[ f_\omega (r) = C \int_{\text{orifice}} f_\omega (r') \frac{e^{ikR}}{R} d^2 r' . \quad (8.71) \]

Here \( f \) is any transverse component of any of wave’s fields (either E or H), \( R \) is the distance between point \( r' \) at the orifice and the observation point \( r \) (i.e. the magnitude of vector \( R \equiv r - r' \)), and \( C \) is a complex constant.

Before describing the proof of Eq. (71), let me carry out its sanity check - which also will give us the constant \( C \). Let us see what happens if the field under the integral is the usual plane wave \( f_\omega(z) \) propagating along axis \( z \) (i.e. there is no opaque screen at all), so we should take the whole \( x-y \) plane, say with \( z' = 0 \), as the integration area (Fig. 10).

Then, for the observation point with coordinates \( x = 0, y = 0, \) and \( z >> \lambda \), Eq. (71) yields

\[ f_\omega (z) = C f_\omega (0) \int dx' \int dy' \exp \left\{ \frac{ik}{2} \left( \frac{x'^2 + y'^2 + z^2}{2} \right)^{1/2} \right\} . \quad (8.72) \]

Before specifying the integration limits, let us consider the range \( |x'|, |y'| << z \). In this range the square root, met in Eq. (72) twice, may be approximated as

\[ \sqrt{\left( x'^2 + y'^2 + z^2 \right)} \approx \sqrt{2z} \left( 1 + \frac{x'^2 + y'^2}{z^2} \right)^{1/2} \approx \frac{x'^2 + y'^2}{2z} . \quad (8.73) \]

The denominator of Eq. (72) is a much slower function of \( x' \) and \( y' \) than the exponent, and in it (as we will check \textit{a posteriori}), it is sufficient to keep just the main, first term of expansion (73). With that, Eq. (72) becomes

\[ \text{The fact that the Huygens principle is valid for any field component should not too surprising. Due to condition } a >> \lambda, \text{ the real boundary conditions at the orifice edges are not important; what is only important that the screen, that limits the orifice, is opaque. Because of this, the Huygens principle’s expression (71) is a part of the so-called scalar theory of diffraction. (In this course I will not have time to go beyond this approximation.)} \]
\[ f_\omega(z) = C f_\omega(0) \frac{e^{ikz}}{z} \int dx' \int dy' \exp \left( \frac{ik(x'^2 + y'^2)}{2z} \right) = C f_\omega(0) \frac{e^{ikz}}{z} I_x I_y, \]  
(8.74)

where \( I_x \) and \( I_y \) are two similar integrals; for example,

\[ I_x = \int \exp \left( \frac{ikx'^2}{2z} \right) dx' = \left( \frac{2z}{k} \right)^{1/2} \int \exp \left( \frac{i\xi^2}{2} \right) d\xi = \left( \frac{2z}{k} \right)^{1/2} \left[ \cos \left( \frac{\xi^2}{2} \right) d\xi + i \int \sin \left( \frac{\xi^2}{2} \right) d\xi \right], \]
(8.75)

where \( \xi \equiv (k/2z)^{1/2} x' \). These are the so-called Fresnel integrals. I will discuss them in more detail in the next section, and right now, only one property of these integrals is important for us: if taken in symmetric limits \([-\xi_0, +\xi_0]\), both of them rapidly converge to the same value, \((\pi/2)^{1/2}\), as soon as \(\xi_0\) becomes much larger than 1.  

This means that even if we do not impose any exact limits on the integration area in Eq. (72), this integral converges to value \(f_\omega(0)e^{ikz}\), due to contributions from the central area with linear size of the order of \(\Delta \xi \sim 1\), i.e.

\[ \Delta x \sim \Delta y \sim \left( \frac{z}{k} \right)^{1/2} \sim (\lambda z)^{1/2}, \]
(8.77)

so that the contribution by front points \(r'\) well beyond the range (77) is negligible.  

(Within our assumptions (70), which in particular require \(\lambda\) to be much less than \(z\), the diffraction angle \(\Delta x/z \sim \Delta y/z \sim (\lambda/z)^{1/2}\), corresponding to the important area of the front, is small.) In order to sustain the plane wave propagation, \(f_\omega(z) = f_\omega(0)e^{ikz}\), constant \(C\) in Eq. (76) has to be taken equal to \(k/2\pi i\). Thus, the Huygens principle’s prediction (71), in its final form, reads

\[ f_\omega(r) = \frac{k}{2\pi i} \int_{\text{orifice}} f_\omega(r') \frac{e^{ikR}}{R} d^2r', \]
(8.78)

and describes, in particular, the straight propagation of the plane wave (in a uniform media).

Let me pause to emphasize how nontrivial this result is. It would be a natural corollary of Eq. (25) (and the linear superposition principle) if all points of the orifice were filled with point scatterers that re-emit all the incident waves into spherical waves. However, as it follows from the above proof, the Huygens principle is also valid if there is nothing in the orifice but the free space!

This is why it is important a proof of the principle, based on the Green’s theorem (2.207). Let us apply this theorem to function \(f = f_\omega\), where \(f_\omega\) is the complex amplitude of a scalar component of one of wave’s fields, which satisfies the Helmholtz equation (7.192),

\[ \text{(8.78)} \]

---

25 See, e.g., MA Eq. (6.10).
26 This result very natural, because \(\exp \{ikR\}\) oscillates fast with the change of \(r'\), so that the contributions from various front point are averaged out. Indeed, the only reason why the central part of plane \([x', y']\) gives a nonvanishing contribution (76) to \(f_\omega(z)\) is that the phase exponents stops oscillating at \((x'^2 + y'^2) \sim z/k\) – see Eq. (73).
27 This proof was given in 1882 by G. Kirchhoff.
\[
\left( \nabla^2 + k^2 \right) f_\omega(r) = 0, \tag{8.79}
\]
and function \( g = g_\omega \), which is the time Fourier image of the corresponding Green’s function. It may be defined, as usual, as the solution to the same equation with the added delta-functional right-hand part with an arbitrary coefficient, for example,

\[
\left( \nabla^2 + k^2 \right) g_\omega(r, r') = -4\pi \delta(r - r'). \tag{8.80}
\]

With Eqs. (79) and (80) used to express the Laplace operators of functions \( f_\omega \) and \( g_\omega \), Eq. (2.207) becomes

\[
\int_V \left\{ f_\omega [-k^2 g_\omega(r, r') - 4\pi \delta(r - r')] - g_\omega(r, r') [-k^2 f_\omega] \right\} d^3r = \oint_S \left[ f_\omega \frac{\partial g_\omega(r, r')}{\partial n'} - g_\omega(r, r') \frac{\partial f_\omega}{\partial n'} \right] d^2r', \tag{8.81}
\]

where \( n \) is the outward normal to the surface \( S \) limiting volume \( V \). Two terms in the left-hand side of this relation cancel, so that after swapping \( r \) and \( r' \) we get

\[
-4\pi f_\omega(r) = \oint_S \left[ f_\omega(r') \frac{\partial g_\omega(r', r)}{\partial n'} - g_\omega(r', r) \frac{\partial f_\omega}{\partial n'} \right] d^2r'. \tag{8.82}
\]

This relation is only correct if the selected volume \( V \) includes point \( r \) (otherwise we would not get its left-hand part from the integration of the delta-function), but does not include the genuine source of the wave (otherwise Eq. (79) would have a nonvanishing right-hand part). Let \( r \) be the field observation point, \( V \) all the source-free half-space (for example, the half-space right of the screen in Fig. 9), so that \( S \) is the surface of the screen, including the orifice. Then the right-hand part of Eq. (82) describes the field in the observation point \( r \) induced by the wave passing through the orifice points \( r' \). Since no waves are emitted by the opaque parts of the screen, we can limit the integration by the orifice area.\(^{28}\) Assuming also that the opaque parts of the screen do not re-emit waves “radiated” by the orifice, we can take the solution of Eq. (80) to be the retarded potential for the free space:\(^{29}\)

\[
g_\omega(r, r') = \frac{e^{iR}}{R}. \tag{8.83}
\]

Plugging this expression into Eq. (82), we get

\[
-4\pi f_\omega(r) = \int_{\text{orifice}} \left[ f_\omega(r') \frac{e^{iR}}{R} - \frac{e^{iR}}{R} \frac{\partial f_\omega(r')}{\partial n'} \right] d^2r'. \tag{8.84}
\]

This is the so-called Kirchhoff (or “Fresnel-Kirchhoff”) integral.\(^{30}\) Now, let us make the two additional approximations. The first of them stems from Eq. (70): at \( ka >> 1 \), the wave’s spatial dependence in the orifice area may be presented as

---

\(^{28}\) Actually, this is a somewhat nontrivial point of the proof. Indeed, it may be shown that the solution of Eq. (79) identically equals to zero if \( f(r') \) and \( \partial f(r')/\partial n' \) vanish together at any part of the boundary. As a result, building the solution with the account of exact boundary conditions (which is the task of the vector theory of diffraction) is possible but cumbersome. Here we base our solution on the physical intuition.

\(^{29}\) It follows, e.g., from Eq. (16) with a monochromatic source \( q(t) = q_\omega \exp \{-i\omega t\} \), at the value \( q_\omega = 4\pi e \) that fits the right-hand part of Eq. (80).

\(^{30}\) With the integration extended over all boundaries of volume \( V \), this would be an exact result.
where “slow” means a function that changes on the scale of $a$ rather than $\lambda$. If, also, $kR \gg 1$, then the differentiation in Eq. (84) may be, in both instances, limited to the rapidly changing exponents, giving

$$\frac{-4\pi f_\omega(r)}{f_\omega(r)} = \int_A i(k + k_\omega) \cdot n' \frac{e^{ikR}}{R} f(r') d^2r', \quad (8.86)$$

Second, if all observation angles are small, we can take $k \cdot n' \approx k_0 \cdot n' \approx -k$. With that, Eq. (86) is reduced to Eq. (78) expressing the Huygens principle.

It is clear that the principle immediately gives a very simple description of the interference of waves passing through two small holes in the screen. Indeed, if the hole size is negligible in comparison with distance $a$ between them (though still much larger than the wavelength!), Eq. (78) yields

$$f_\omega(r) = c_1 e^{ikR_1} + c_2 e^{ikR_2}, \quad \text{with} \quad c_{1,2} \equiv \frac{kf_{1,2}A_{1,2}}{2\pi R_{1,2}}, \quad (8.87)$$

where $R_{1,2}$ are the distances between the holes and the observation point, and $A_{1,2}$ are the hole areas. For the interference wave intensity, Eq. (87) yields

$$S \propto f_\omega \cdot f_\omega^* = |c_1|^2 + |c_2|^2 + 2|c_1||c_2| \cos[k(R_1 - R_2) + \varphi], \quad \varphi \equiv \arg c_1 - \arg c_2. \quad (8.88)$$

The first two terms in this result clearly represent the intensities of partial waves passed through each hole, while the last one the result of their interference. The interference pattern’s contrast ratio

$$R \equiv \frac{S_{\max}}{S_{\min}} = \left(\frac{|c_1| + |c_2|}{|c_1| - |c_2|}\right)^2, \quad (8.89)$$

is largest (infinite) when both waves have equal amplitudes.

The analysis of the interference pattern is simple if the line connecting the holes is perpendicular to wave vector $k \approx k_0$ – see Fig. 6a. Selecting the coordinate axes as shown in that figure, and using for distances $R_{1,2}$ the same expansion as in Eq. (73), for the interference term in Eq. (88) we get

$$\cos[k(R_1 - R_2) + \varphi] \approx \cos\left(\frac{kx}{z} + \varphi\right). \quad (8.90)$$

This means that the intensity does not depend on $y$, i.e. the interference pattern in the plane of constant $z$ presents straight, parallel strips, perpendicular to vector $a$, with the period given by Eq. (60), i.e. by the Bragg law.\textsuperscript{31} Note that this (somewhat counter-intuitive) result is strictly valid only at $(x^2 + y^2) \ll z^2$; it is straightforward to use the next term in the Taylor expansion (73) to show that farther from the interference pattern center the strips start to diverge.

\textsuperscript{31} The phase shift $\varphi$ vanishes at the normal incidence of a plane wave on the holes. Note, however, that the spatial shift of the interference pattern following from Eq. (90), $\Delta x = -z(ka)\varphi$, is extremely convenient for the experimental measurement of the phase shift between two waves, especially if it is induced by some factor (such as insertion of a transparent object into one of interferometer’s arms, etc.) that may be turned on/off at will.
8.6. Diffraction on a slit

Now let us use the Huygens principle to analyze a more complex problem: plane wave’s diffraction on a long straight slit of constant width \( a \) (Fig. 11).

According to Eq. (70), in order to use the Huygens principle for the problem analysis we need to have \( \lambda << a << z \). Moreover, the simple formulation (78) of the principle is only valid for small observation angles, \( |x| << z \). Note, however, that the relation between two small dimensionless numbers, \( z/a \) and \( a/\lambda \) is so far arbitrary; as we will see in a minute, this relation will determine the type of the observed diffraction pattern.

Let us apply Eq. (78) to our current problem (Fig. 11), for the sake of simplicity assuming the normal wave incidence, and taking \( z = 0 \) at the screen plane:

\[
f_{\omega}(x,z) = f_0 \frac{k}{2\pi i} \int_{-a/2}^{+a/2} dx' \int_{-\infty}^{+\infty} dy' \exp \left( ik \frac{(x-x')^2 + y'^2 + z^2}{2z} \right),
\]

where \( f_0 \equiv f_\omega(x',0) = \text{const} \) is the incident wave’s amplitude. This is the same integral as in Eq. (72), except for the finite limits for \( x' \), and may be simplified similarly, using the small-angle condition \((x - x')^2 + y'^2 << z^2\):

\[
f(x,z) \approx f_0 \frac{k}{2\pi i} \frac{e^{ikz}}{z} \int_{-a/2}^{+a/2} dx' \int_{-\infty}^{+\infty} dy' \exp \left( ik \frac{(x-x')^2 + y'^2}{2z} \right) = f_0 \frac{k}{2\pi i} \frac{e^{ikz}}{z} I_x I_y.
\]

The integral over \( y \) is the same as in the last section:

\[
I_y \equiv \int_{-\infty}^{+\infty} \exp \left( \frac{iky'^2}{2z} \right) dy' = \left( \frac{2\pi iz}{k} \right)^{1/2}
\]

but the integral over \( x \) is more complicated, because of its finite limits:

\[
I_x \equiv \int_{-a/2}^{+a/2} \exp \left( \frac{ik(x-x')^2}{2z} \right) dx'.
\]

It may be simplified in the following two (opposite) limits.
(i) **Fraunhofer diffraction** takes place when \( z/a >> a/\lambda \) - the relation which may be rewritten either as \( a << (z\lambda)^{1/2} \), or as \( ka^2 << z \). In this limit the ratio \( kx'^2/z \) is negligibly small for all values of \( x' \) under the integral, and we can approximate it as

\[
\frac{2\sin^2 e^{ix'^2/2}}{2z} \approx \frac{2\sin^2 e^{i\eta x'^2/2}}{z},
\]

so that Eq. (92) yields

\[
f_o(x, z) \approx \frac{k}{2 \pi i} \frac{e^{ikz}}{z} \left( \frac{2\pi i z}{k x} \right)^{1/2} \exp \left( -\frac{ikx'^2}{2z} \right), \tag{8.96}
\]

and hence the relative wave intensity is

\[
\frac{S(x, z)}{S_0} = \left| \frac{f_o(x, z)}{f_0} \right|^2 = \frac{8z}{\pi k^2} \sin^2 \left( \frac{kxa}{2z} \right) = \frac{2}{\pi} \frac{ka^2}{z} \sin^2 \left( \frac{ka\alpha}{2} \right), \tag{8.97}
\]

where \( S_0 \) is the (average) intensity of the incident wave, and \( \alpha \equiv x/z << 1 \) is the scattering angle. Comparing this expression with Eq. (69), we see that this the diffraction pattern is exactly the same as that of a similar (uniform, 1D) object in the Born approximation – see the red line in Fig. 8. Note again that the angular width \( \delta\alpha \) of the Fraunhofer pattern is of the order of \( 1/ka \), so that its linear width \( \delta x = z\delta\alpha ~ z/ka ~ z\lambda/a \).\(^{32}\) Hence the condition of the Fraunhofer approximation validity may be also presented as \( a << \delta x \).

(ii) **Fresnel diffraction.** In the opposite limit of a relatively wide slit, with \( a >> \delta x = z\delta\alpha ~ z/ka ~ z\lambda/a \), i.e. \( ka^2 >> z \), the diffraction patterns at two slit edges are well separated. Hence, near each edge (for example, near \( x' = -a/2 \)) we may simplify Eq. (94) as

\[
I(x) \approx \int_{-a/2}^{+\infty} \frac{2z}{k} \exp \left( \frac{ik(x-x')^2}{2z} \right) dx' = \left( \frac{2z}{k} \right)^{1/2} \int_{(k/2z)^{1/2}/2}^{+\infty} \exp \left( \frac{\zeta^2}{2(x+a/2)} \right) d\zeta', \tag{8.98}
\]

and express it via the special functions called the **Fresnel integrals**;\(^{33}\)

\[
\varphi(\xi) \equiv \left( \frac{2}{\pi} \right)^{1/2} \int_0^{\xi/2} \cos(\zeta^2) d\zeta', \quad \varphi(\xi) \equiv \left( \frac{2}{\pi} \right)^{1/2} \int_0^{\xi/2} \sin(\zeta^2) d\zeta', \tag{8.99}
\]

whose plots are shown in Fig. 12. As was mentioned above, at large values of their argument \( \xi \), both functions tend to \( 1/2 \).

\(^{32}\) Note also that since in this limit \( ka^2 << z \), Eq. (97) shows that even the maximum value \( S(0, z) \) of the diffracted wave intensity is much less than intensity \( S_0 \) of the incident wave. This is natural, because the incident power \( S_0 a \) per unit length of the slit is now distributed over a much larger width \( \delta x >> a \), so that \( S(0, z) ~ S_0 (a/\delta x) << S_0 \).

\(^{33}\) Slightly different definitions of these functions, mostly affecting constant factors, may also be met in literature.
Plugging this expression into Eq. (92) and (98), for the diffracted wave intensity, in the Fresnel limit (i.e. at $|x + a/2| \ll a$), we get

$$\bar{S}(x, z) = \frac{1}{2}\left[\varphi\left(\frac{k}{2z}\left(x + \frac{a}{2}\right) + \frac{1}{2}\right)\right]^2 + \left[S\left(\frac{1}{2}\left(x + \frac{a}{2}\right) + \frac{1}{2}\right)\right]^2.$$  \hspace{0.5cm} (8.100)

A plot of this function (Fig. 13) shows that the diffraction pattern is very peculiar: while in the “shade” region $x < -a/2$ the wave intensity fades monotonically, the transition to the “light” region within the gap ($x > -a/2$) is accompanied by intensity oscillations, just as at the Fraunhofer diffraction – cf. Fig. 8.

This behavior, which is described by the following asymptotes,

$$\frac{\bar{S}}{S_0} \rightarrow \begin{cases} 1 + \frac{1}{\sqrt{\pi}} \frac{\sin(\xi^2 - \pi/4)}{\xi}, & \text{for } \xi = \left(\frac{k}{2z}\right)^{1/2}\left(x + \frac{a}{2}\right) \rightarrow +\infty, \\ \frac{1}{4\pi\xi^2}, & \text{for } \xi \rightarrow -\infty, \end{cases}$$  \hspace{0.5cm} (8.101)

is essentially an artifact of observing just the wave intensity (i.e. its real amplitude) rather than its phase as well. Indeed, as may be seen even more clearly from the parametric presentation of the Fresnel
integrals (Fig. 14), these functions oscillate similarly at large positive and negative values of their argument. Physically, this means that the wave diffraction by the slit edge leads to similar oscillations of its phase at \( x < -a/2 \) and \( x > -a/2 \); however, in the latter region (i.e. inside the slit) the diffracted wave overlaps the incident wave passing through the slit directly, and their interference reveals the phase oscillations, making them visible in the measured intensity as well.

![Fresnel integrals](image)

Fig. 8.14. Parametric representation of the Fresnel integrals. This pattern is called either the Euler spiral or Cornu spiral.

Note that according to Eq. (100), the linear scale of the Fresnel diffraction pattern is \( (2z/k)^{1/2} \), i.e. is complied with estimate (77). If the slit is gradually narrowed, so that width \( a \) becomes comparable to that scale,\(^{34}\) the Fresnel interference patterns from both edges start to “collide” (interfere). The resulting wave, fully described by Eq. (94), is just a sum of two contributions of the type (98) from the both edges of the slit. The resulting interference pattern is somewhat complicated, and only \( a << \delta \xi \) it is reduced to the simple Fraunhofer pattern (97). Of course, this crossover from the Fresnel to Fraunhofer diffraction may be also observed, at fixed wavelength \( \lambda \) and slit width \( a \), by increasing \( z \), i.e. by measuring the diffraction pattern farther and farther from the slit.

Note that the Fraunhofer limit is always valid if the diffraction measured as a function of the diffraction angle \( \alpha \) alone, i.e. effectively at infinity, \( z \rightarrow \infty \). This may be done, for example, by collecting the diffracted wave with a “positive” (converging) lense, and observing the diffraction pattern in its focal plane.

8.7. Geometrical optics placeholder

Behind all these details, I would not like the reader to miss the main feature of diffraction, that has an overwhelming practical significance. Namely, besides narrow diffraction “cones” (actually, parabolic-shaped regions) with lateral scale \( \Delta x \sim (\lambda z)^{1/2} \), the wave far behind a slit of width \( a >> \lambda \) repeats the field just behind the slit, i.e. reproduces the unperturbed incident wave inside the slit, and has negligible intensity in the shade regions outside it. An evident generalization of this fact is that when a plane wave (in particular an electromagnetic wave) passes any opaque object of large size \( a >> \lambda \), it propagates around it, by distances \( z \) up to \( \sim a^2/\lambda \), along straight lines, with virtually negligible diffraction

\(^{34}\) Note that this condition may be also rewritten as \( a \sim \delta \xi \), i.e. \( z/a \sim a/\lambda \).
effects. This fact gives the strict foundation for the very notion of the wave ray (or beam), as the line perpendicular to the local front of a quasi-plane wave. In a uniform media such ray is a straight line, but changes in accordance with the Snell law at the interface of two media with different wave speed \( v \), i.e. different values of the refraction index. The notion of rays enables the whole field of geometric optics, devoted mostly to ray tracing in various (sometimes very complex) systems.

This is why, at this point, an E&M course that followed the scientific logic more faithfully than this one, would give an extended discussion of the geometric and quasi-geometric optics, including (as a minimum) such vital topics as

- the so-called lensmaker’s equation expressing the focus length \( f \) of a lens via the curvature radii of its spherical surfaces and the refraction index of the lens material,
- the thin lens formula relating the image distance from the lens via \( f \) and the source distance,
- the concepts of basic optical instruments such as telescopes and microscopes,
- the concepts of the spherical, angular, and chromatic aberrations (image distortions);
- wave effects in optical instruments, including the so-called Abbe limit on the focal spot size.

However, since I have made a (possibly, wrong) decision to follow the common tradition in selecting the main topics for this course, I do not have time left for such discussion. Still, I am placing this “placeholder” pseudo-section to relay my conviction that any educated physicist has to know the geometric optics basics. If the reader has not had an exposure to this subject during his or her undergraduate studies, I highly recommend at least browsing one of available textbooks.

8.8. Fraunhofer diffraction from more complex scatterers

So far, our discussion of diffraction has been limited to a very simple geometry – a single slit in an otherwise opaque screen (Fig. 11). However, in the most important Fraunhofer limit, \( z \gg ka^2 \), it is easy to get a very simple expression for the plane wave diffraction/interference by a plane orifice (with linear size \( \sim a \)) of an arbitrary shape. Indeed, the evident 2D generalization of approximation (93)-(94) is

\[
I_x I_y = \int_{\text{orifice}} \frac{\exp \left[ ik \left( (x-x')^2 + (y-y')^2 \right) \right]}{2z} \, dx'dy' \\
\approx \exp \left\{ ik \frac{(x^2 + y^2)}{2z} \right\} \int_{\text{orifice}} \exp \left\{ -i \frac{kx'}{z} - i \frac{ky'}{z} \right\} dx'dy',
\]

(8.102)

\[ 35 \text{ Admittedly, even this list leaves aside several spectacular effects due to crystal anisotropy, including such a beauty as conical refraction in biaxial crystals - see, e.g., Chapter 15 of the classical textbook by M. Born and E. Wolf, cited in the end of Sec. 7.1.} \]

\[ 36 \text{ Reportedly, due not only E. Abbe (1873), but also to H. von Helmholtz (1874).} \]

\[ 37 \text{ In contrast to other topics of this list, whose study may be based on the ray approach, i.e. on purely geometric optics, the description of these effects requires at least an approximate account of wave properties of light. Such account may be based either on the Huygens principle or on the so-called paraxial equation} \]

\[
\frac{\partial a}{\partial z} = \left( 1/2ik \right) \nabla^2_{x,y} a,
\]

for the complex amplitude \( a(\mathbf{r}) \) of the field represented in the form \( f(\mathbf{r}) = a(\mathbf{r}) e^{ikz} \). The paraxial approximation follows from the Helmholtz equation (7.192) in essentially the same limit \( (|\nabla a| \ll k; |x|, |y| \ll z) \) as Eq. (78).

\[ 38 \text{ My top recommendation for that purpose would be Chapters 3-6 and Sec. 8.6 in Born and Wolf. A simpler alternative is Chapter 10 in G. R. Fowles, } \]

so that besides the inconsequential total phase factor, Eq. (92) is reduced to

$$ f(\rho) \propto f_0 \int \exp[-i\kappa \cdot \rho'] d^2 \rho' = f_0 \int T(\rho') \exp[-i\kappa \cdot \rho'] d^2 \rho', $$

(8.103)

where the 2D vector $\kappa$ (not to be confused with wave vector $k$ that is virtually perpendicular to $\kappa$!) is defined as

$$ \kappa \equiv \frac{k}{z} \approx q \equiv k - k_0, $$

(8.104)

$$ \rho = \{x, y\} \text{ and } \rho' = \{x', y'\} \text{ are 2D radius-vectors in, respectively, the observation and screen planes (both nearly normal to vectors } k \text{ and } k_0, \text{ function } T(\rho') \text{ describes screen's transparency at point } \rho', \text{ and the last integral in Eq. (103) is over the whole screen plane } z' = 0. (\text{Though the strict equivalence of the two forms of Eq. (103) is only valid if } T(\rho') = \text{either 1 or 0, its last form may be readily obtained from Eq. (78) with } f(r') = T(\rho') f_0 \text{ for any transparency profile, provided that } T(\rho') \text{ is an arbitrary function but changes only at distances much larger than } \lambda \equiv 2\pi/k.) $$

From the mathematical point of view, the last form of Eq. (103) is the 2D spatial Fourier transform of function $T(\rho')$, with the reciprocal variable $\kappa$ revealed by the observation point position: $\rho = (z/k)\kappa = (z\lambda/2\pi)\kappa$. This interpretation is useful because of the experience we all have with the Fourier transform, mostly in the context of its time/frequency applications. For example, if the orifice is a single small hole, $T(\rho')$ may be approximated by a delta-function, so that Eq. (103) yields $f(\rho) \approx \text{const.}$ This corresponds (at least for the small diffraction angles $\alpha \equiv \rho'z$, for which the Huygens approximation is valid) to a spherical wave spreading from the point-like orifice. Next, for two small holes, Eq. (103) immediately gives the Young interference pattern (90). Let me now use Eq. (103) to analyze the simplest (and most important) 1D transparency profiles, leaving 2D cases for reader’s exercise.

(i) A single slit of width $a$ (Fig. 11) may be described by transparency

$$ T(\rho') = \begin{cases} 1, & \text{for } |x'| < a/2, \\ 0, & \text{otherwise.} \end{cases} $$

(8.105)

Its substitution into Eq. (103) yields

$$ f(\rho) \propto f_0 \int_{-a/2}^{+a/2} \exp[-i\kappa_s x'] dx' = f_0 \frac{\exp[-i\kappa_s a/2] - \exp[i\kappa_s a/2]}{-i\kappa_s} \alpha \sin \left( \frac{\kappa_s a}{2} \right) = \sin \left( \frac{kx}{2z} \right), $$

(8.106)

naturally returning us to Eqs. (64) and (97), and hence to the red lines in Fig. 8 for the wave intensity. (Please note again that Eq. (103) describes only the Fraunhofer, but not the Fresnel diffraction!)

(ii) Two narrow similar, parallel slits with a much larger distance $a$ between them, may be described by taking

$$ T(\rho') \propto \delta(x' - a/2) + \delta(x' + a/2), $$

(8.107)

so that Eq. (103) yields the generic interference pattern,

$$ f(\rho) \propto f_0 \left[ \exp \left( -\frac{i\kappa_s a}{2} \right) + \exp \left( \frac{i\kappa_s a}{2} \right) \right] \propto \cos \left( \frac{\kappa_s a}{2} \right) = \cos \left( \frac{kx}{2z} \right), $$

(8.108)
whose intensity is shown with the blue line in Fig. 8.

(iii) In a more realistic Young-type two-slit experiment, each slit has width (say, \( w \)) which is much larger than light wavelength \( \lambda \), but still much smaller than slit spacing \( a \). This situation may be described by the following transparency function

\[
T(p') = \sum_{\pm} \begin{cases} 
1, & \text{for } |x' + a/2| < w/2, \\
0, & \text{otherwise},
\end{cases}
\]

for which Eq. (103) yields a natural combination of results (106) (with \( a \) replaced with \( w \)) and (108):

\[
f(r) \propto \text{sinc}\left(\frac{kxw}{2z}\right) \cos\left(\frac{kxa}{2z}\right).
\]

This is the usual interference pattern modulated by a Fraunhofer-diffraction envelope (shown with the dashed blue line Fig. 15). Since function \( \text{sinc}^2 \xi \) decreases very fast beyond its first zeros at \( \xi = \pm \pi \), the practical number of observable interference fringes is close to \( 2a/w \).

(iv) A structure very useful for experimental and engineering practice is a set of many parallel slits, called the diffraction grating.\(^{39}\) Indeed, if the slit width is much less than the grating period \( d \), then the transparency function may be approximated as

\[
T(p') \propto \sum_{n=-\infty}^{\infty} \delta(x' - nd),
\]

and Eq. (103) yields

\[
f(p) \propto \sum_{n=-\infty}^{\infty} \exp\{-in\kappa_d d\} = \sum_{n=-\infty}^{\infty} \exp\left\{-i\frac{nkd}{z}\right\}.
\]

This sum vanishes for all values of \( \kappa_d d \) that are not multiples of \( 2\pi \), so that the result describes sharp intensity peaks at diffraction angles

---

\(^{39}\) The rudimentary diffraction grating effect, produced by parallel fibers of bird feathers, was discovered as early as in 1673 by J. Gregory - who has also invented the reflecting (“Gregorian”) telescope.
Taking into account that this result is only valid for small angles $|\alpha_m| \ll 1$, it may be interpreted exactly as Eq. (59) – see Fig. 6a. However, in contrast with the interference (108) from two slits, the destructive interference from many slits kills the net wave as soon as the angle is even slightly different from each Bragg angle (60). This is very convenient for spectroscopic purposes, because the diffraction lines produced by multi-frequency waves do not overlap even if the frequencies of their adjacent components are very close.

Two features of practical diffraction gratings make their properties different from this simple picture. First, the finite number $N$ of slits, which may be described by limiting sum (112) to interval $n = [-N/2, +N/2]$, results in the finite spread, $\delta \alpha / \alpha \sim 1/N$, of each diffraction peak, and hence in the reduction of grating’s spectral resolution. (Unintentional variations of the inter-slit distance $d$ have a similar effect, so that before the advent of high-resolution photolithography, special high-precision mechanical tools have been used for grating fabrication.)

Second, the finite slit width $w$ leads to the diffraction peak pattern modulation by a $\text{sinc}^2(kw \alpha/2)$ envelope, similarly to pattern shown in Fig. 15. Actually, for spectroscopic purposes such modulation is a plus, because only one diffraction peak (say, with $m = \pm 1$) is practically used, and if the frequency spectrum of the analyzed wave is very broad (cover more than one octave), the higher peaks produce undesirable hindrance. Because of this reason, $w$ is frequently selected to be equal exactly to $d/2$, thus suppressing each other diffraction maximum. Moreover, sometimes semi-transparent films are used to make the transparency function $T(r')$ continuous and close to the sinusoidal one:

$$T(r') \approx T_0 + T_1 \cos \frac{2\pi \alpha'}{d} = T_0 + \frac{T_1}{2} \left\{ \exp \left(i \frac{2\pi \alpha'}{d} \right) + \exp \left(-i \frac{2\pi \alpha'}{d} \right) \right\}. \quad (8.114)$$

Plugging the last expression into Eq. (103) and integrating, we see that the output wave consists of just 3 components: the direct-passing wave (proportional to $T_0$) and two diffracted waves (proportional to $T_1$) propagating in the directions of the two lowest Bragg angles, $\alpha_{\pm 1} = \pm \lambda/d$.

Relation (103) may be also readily used to obtain one more general (and rather curious) result called the Babinet principle. Consider two experiments with diffraction of similar plane waves on two “complementary” screens who together would cover the whole plane, without a hole or an overlap. (Think, for example, about an opaque disk of radius $R$ and a large opaque screen with a round orifice of the same radius.) Then, according to the Babinet principle, the diffracted wave patterns produced by these two screens in all directions with $\alpha \neq 0$ are identical. The proof of this principle is straightforward: since the transparency functions produced by the screens are complementary in the following sense:

$$T(p') \equiv T_1(p') + T_2(p') = 1, \quad (8.115)$$

and (in the Fraunhofer approximation (103) only!) the diffracted wave is a linear Fourier transform of $T(p')$, we get

$$f_1(p) + f_2(p) = f_0(p), \quad (8.116)$$

where $f_0$ is the wave “scattered” by the composite screen with $T_0(p') \equiv 1$, i.e. the unperturbed initial wave propagating in the initial direction ($\alpha = 0$). In all other directions, $f_1 = -f_2$, i.e. the diffracted waves
are indeed similar besides the difference in sign - which is equivalent to a phase shift by \( \pm \pi \). However, it is important to remember that the Babinet principle notwithstanding, in real experiments the diffracted waves may interfere with the unperturbed plane wave \( f_0(\mathbf{r}) \), leading to different diffraction pattern in cases 1 and 2 – see, e.g., Fig. 13 and its discussion.

### 8.9. Magnetic dipole and electric quadrupole radiation

Throughout this chapter, we have seen how many important results may be obtained from Eq. (26) for the electric dipole radiation by a small-size source (Fig. 1). Only in rare cases when such radiation is absent, for example if the dipole moment \( \mathbf{p} \) of the source equals zero (or does not change at time – either at all, or at the frequency of our interest), higher-order effects may be important. I will discuss the main two of them, the quadrupole electric and dipole magnetic radiation – mostly for reference purposes, because we would not have much time to discuss their applications.

In Sec. 2 above, the electric dipole radiation was calculated by plugging the first, leading term of expansion (19) into the exact formula (17b) for the retarded vector-potential \( \mathbf{A}(\mathbf{r}, t) \). Let us make a more exact calculation, by keeping the second term of that expansion as well:

\[
\mathbf{j}\left(\mathbf{r}', t - \frac{R}{v}\right) \approx \mathbf{j}\left(\mathbf{r}', t - \frac{\mathbf{r}' \cdot \mathbf{n}}{v}\right) = \mathbf{j}\left(\mathbf{r}', t' + \frac{\mathbf{r}' \cdot \mathbf{n}}{v}\right), \quad \text{where} \quad t' = t - \frac{\mathbf{r}}{v}.
\]  

(8.117)

Since this expansion is only valid if the last term in the second argument is relatively small, in the Taylor expansion of \( \mathbf{j} \) with respect to that argument we may keep just the first two leading terms:

\[
\mathbf{j}\left(\mathbf{r}', t' + \frac{\mathbf{r}' \cdot \mathbf{n}}{v}\right) \approx \mathbf{j}(\mathbf{r}', t') + \frac{1}{v} \frac{\partial}{\partial t'} \mathbf{j}(\mathbf{r}', t')(\mathbf{r}' \cdot \mathbf{n}),
\]

(8.118)

so that Eq. (17b) yields \( \mathbf{A} = \mathbf{A}_e + \mathbf{A}' \), where \( \mathbf{A}_e \) is the electric dipole contribution as given by Eq. (23), and \( \mathbf{A}' \) is the new term of the next order in small parameter \( \mathbf{r}' \ll \mathbf{r} \):

\[
\mathbf{A}'(\mathbf{r}, t) = \frac{\mu}{4\pi rv} \frac{\partial}{\partial t'} \int \mathbf{j}(\mathbf{r}', t')(\mathbf{r}' \cdot \mathbf{n}) d^3r'.
\]

(8.119)

Just as was done in Sec. 2, let us evaluate this term for a system of non-relativistic particles with electric charges \( q_k \) and radius-vectors \( \mathbf{r}_k(t) \):

\[
\mathbf{A}'(\mathbf{r}, t) = \frac{\mu}{4\pi rv} \left[ \frac{d}{dt} \sum_k q_k \mathbf{r}_k(\mathbf{r}_k \cdot \mathbf{n}) \right]_{t=t'}.
\]

(8.120)

Using the “bac minus cab” identity of the vector algebra again,\(^{40}\) Eq. (120) may be rewritten as

\[
\mathbf{r}_k(\mathbf{r}_k \cdot \mathbf{n}) = \frac{1}{2} \mathbf{r}_k(\mathbf{r}_k \cdot \mathbf{n}) + \frac{1}{2} \mathbf{r}_k(\mathbf{n} \cdot \mathbf{r}_k) = \frac{1}{2} (\mathbf{r}_k \times \mathbf{r}_k) \times \mathbf{n} + \frac{1}{2} \mathbf{r}_k(\mathbf{n} \cdot \mathbf{r}_k) + \frac{1}{2} \mathbf{r}_k(\mathbf{n} \cdot \mathbf{r}_k)
\]

(8.121)

\[
= \frac{1}{2} (\mathbf{r}_k \times \mathbf{r}_k) \times \mathbf{n} + \frac{1}{2} \frac{d}{dt} \left[ \mathbf{r}_k(\mathbf{n} \cdot \mathbf{r}_k) \right],
\]

so that the right-hand part of Eq. (120) may be presented as a sum of two terms, \( \mathbf{A}' = \mathbf{A}_m + \mathbf{A}_q \), where

\(^{40}\) If you need, see, e.g., MA Eq. (7.5).
\[
A_m (r, t) = \frac{\mu}{4\pi rv} \mathbf{m}(t) \times \mathbf{n} = \frac{\mu}{4\pi rv} \mathbf{m} \Big( t - \frac{r}{v} \Big) \times \mathbf{n}, \quad \text{with } \mathbf{m}(t) = \frac{1}{2} \sum_k \mathbf{r}_k(t) \times \mathbf{q}_k \mathbf{r}_k(t),
\] 
\[
A_q (r, t) = \frac{\mu}{8\pi rv} \left[ \frac{d^2}{dt^2} \sum_k \mathbf{q}_k \mathbf{r}_k \cdot \mathbf{n} \right] \right]_{t=t}.
\] 

Comparing the second of Eqs. (122) with Eq. (5.91), we see that \( \mathbf{m} \) is just the magnetic moment of the source. On the other hand, the first of Eqs. (122) is absolutely similar in structure to Eq. (23), with \( \mathbf{p} \) replaced by \((\mathbf{m} \times \mathbf{n})/v\), so that for the corresponding component of the magnetic field it gives (in the same approximation \( r >> \lambda \)) the result similar to Eq. (24):

\[
B_m (r, t) = \frac{\mu}{4\pi rv^2} \mathbf{n} \times \left[ \mathbf{v} - \frac{r}{v} \mathbf{m} \right] = -\frac{\mu}{4\pi rv^2} \mathbf{n} \times \left[ \mathbf{v} - \frac{r}{v} \mathbf{m} \right].
\] 

According to this expression, just as at the electric dipole radiation, vector \( \mathbf{B} \) is perpendicular to vector \( \mathbf{n} \), and its magnitude is also proportional to the \( \sin \theta \), where \( \theta \) is now the angle between the direction toward the observation point and the second time derivative of vector \( \mathbf{m} \) rather than \( \mathbf{p} \):

\[
B_m = \frac{\mu}{4\pi rv^2} \mathbf{v} \sin \theta.
\] 

As the result, the intensity of this magnetic dipole radiation has the similar angular distribution:

\[
S_r = ZH^2 = \frac{Z}{(4\pi v^2 r)^2} \left[ \mathbf{v} \left( t - \frac{r}{v} \right) \right]^2 \sin^2 \theta
\] 

- cf. Eq. (26). Note, however, that this radiation is usually much weaker than its electric counterpart. For example, for a non-relativistic particle with electric charge \( q \), moving on a trajectory with of size \( \sim a \), the electric dipole moment is of the order of \( qa \), while its magnetic moment scales as \( qa^2 \omega \), where \( \omega \) is the motion frequency. As a result, the ratio of the magnetic and electric dipole radiation intensities is of the order of \((a^2v^2)^2\), i.e. the squared ratio of particle’s speed to the speed of emitted waves – that has to be much smaller than 1 for our non-relativistic estimate to be valid.

The angular distribution of the electric quadrupole radiation, described by Eq. (123), is more complicated. In order to show this, we may add to \( \mathbf{A}_q \) a vector parallel to \( \mathbf{n} \) (i.e. along the wave propagation), getting

\[
\mathbf{A}_q (r, t) \rightarrow \frac{\mu}{24\pi rv} \mathbf{Q} \left( t - \frac{r}{v} \right), \quad \text{where } \mathbf{Q} \equiv \sum_k \{3 \mathbf{r}_k \mathbf{r}_k \mathbf{r}_k \mathbf{r}_k - \mathbf{n} r_k^2 \},
\] 

because this addition does not give any contribution to the transverse component of the electric and magnetic fields, i.e. to the radiated wave. According to the above definition of vector \( \mathbf{Q} \), its Cartesian components may be presented as

\[41\text{ In electrostatics, the symmetric, zero-trace tensor } Q \text{ determines the next term in the potential expansion (3.5):}
\]

\[
\phi(\mathbf{r}) = \frac{1}{4\pi \varepsilon_0} \left( \frac{1}{r} Q + \frac{1}{r^2} \sum_{j=1}^{3} n_j p_j + \frac{1}{2r^2} \sum_{j,j'=1}^{3} n_j n_j Q_{jj'} + \ldots \right).
\]
\[ Q_j = \sum_{j'=1}^{3} Q_{jj'} n_{j'} , \quad (8.128) \]

where \( Q_{jj'} \) are elements of the so-called electric quadrupole tensor \( Q \) of the system:\(^{42}\)

\[ Q_{jj'} = \sum_k q_k \left( 3 r_j r_{j'} - r^2 \delta_{jj'} \right)_k . \quad (8.129a) \]

For clarity, let me spell out the tensor in its matrix form:

\[
\begin{bmatrix}
2x^2 - y^2 - z^2 & 3xy & 3xz \\
3xy & 2y^2 - x^2 - z^2 & 3yz \\
3xz & 3yz & 2z^2 - x^2 - y^2 \\
\end{bmatrix}_k . \quad (8.129b)
\]

Differentiating the first of Eqs. (127) at \( r >> \lambda \), we get

\[
B_n (\mathbf{r}, t) = -\frac{\mu}{24\pi r v^2} n \times \mathbf{Q} \left( t - \frac{r}{v} \right) . \quad (8.130)
\]

Superficially, this expression is similar to Eqs. (24) or (124), but according to Eqs. (127) and (129), components of vector \( \mathbf{Q} \) depend on the direction of vector \( \mathbf{n} \), leading to a different angular dependence of \( S_r \).

As the simplest example, let us consider a system of two equal point electric charges moving at equal distances \( d(t) \ll \lambda \) from a stationary center (Fig. 16).

![Fig. 8.16. The simplest system emitting electric quadrupole radiation.](image)

Due to the symmetry of the system, its dipole moments \( \mathbf{p} \) and \( \mathbf{m} \) (and hence its electric and magnetic dipole radiation) vanish, but the quadrupole tensor (129) still has nonvanishing components. With the coordinate choice shown in Fig. 16, these components are diagonal:

\[
Q_{xx} = Q_{yy} = -2qd^2 , \quad Q_{zz} = 4qd^2 . \quad (8.131)
\]

With axis \( x \) in the plane of the direction \( \mathbf{n} \) toward the source (Fig. 16), so that \( n_x = \sin \theta, n_y = 0, n_z = \cos \theta \), Eq. (128) yields

\[
Q_x = -2qd^2 \sin \theta , \quad Q_y = 0 , \quad Q_z = 4qd^2 \cos \theta , \quad (8.132)
\]

\(^{42}\) As a math reminder, tensor is a matrix describing a physical reality independent of the reference frame choice, so that the Cartesian elements of the tensor have to change according to certain geometric rules if the reference frame is changed - e.g., rotated. This notion is very similar to a physical vector, that may be described by an ordered set of its Cartesian components, which change according to certain rules as the result of the reference frame’ change. We may be confident that a matrix represents a tensor if it provides a linear relation between components of two physical vectors – such a \( \mathbf{Q} \) and \( \mathbf{n} \) in Eq. (128).
so that the vector product in Eq. (130) has only one nonvanishing Cartesian component:

$$\mathbf{n} \times \mathbf{Q} = n_x \mathbf{\hat{Q}_x} = -n_z \mathbf{\hat{Q}_z} = -6q \sin \theta \cos \theta \frac{d^2(t)}{dt^2} \left[ d^2(t) \right].$$

(8.133)

As a result, the radiation intensity is proportional to $\sin^2 \theta \cos^2 \theta$, i.e., vanishes not only along the symmetry axis (as the dipole radiation does), but also in all directions perpendicular to this axis, reaching its maximum at $\theta = \pi/4$.

For more complex systems, the angular distribution of the electric quadrupole radiation may be different, but its total power may be always presented in a simple form

$$P_q = \frac{Z}{1440 \pi^4} \sum_{j,j'=1}^{3} (\mathbf{\hat{j}} \cdot \mathbf{\hat{j}'})^2. \quad (8.134)$$

Let me finish this section by giving, without proof, one more fact important for applications: due to their different spatial structure, the magnetic dipole and electric quadrupole radiation fields do not interfere, i.e., the total power of radiation (neglecting higher multipole terms) may be found as the sum of these components, calculated independently.

### 8.10. Exercise problems

8.1.* In the electric dipole approximation, calculate the angular distribution and total power of electromagnetic radiation by the following classical model of the hydrogen atom: an electron rotating, at a constant distance $r$, about a much heavier proton. Use the latter result to evaluate the classical lifetime of the atom, borrowing the initial value of $R$ from quantum mechanics: $R(0) = r_B \approx 0.53 \times 10^{-10}$ m.

8.2. A non-relativistic particle of mass $m$ with the electric charge $q$ is placed into a uniform magnetic field $\mathbf{B}$. Derive the law of decrease of particle’s kinetic energy due to its electromagnetic radiation at the cyclotron frequency $\omega_c = qB/m$. Evaluate the rate of such radiation cooling for electrons in a magnetic field of 1 T, and estimate the electron energy interval in which this result is qualitatively correct.

*Hint:* The cyclotron motion will be discussed in detail (for arbitrary particle velocities $v \ll c$) in Sec. 9.6 below, but I hope that the reader knows that in the non-relativistic case ($v \ll c$) the above formula for $\omega_c$ may be readily obtained by combining the 2nd Newton law $mv_\perp^2/R = qv_\perp B$ for the circular motion of the particle under the effect of the magnetic component of the Lorentz force (5.10), and the geometric relation $v_\perp = R\omega_c$. (Here $v_\perp$ is particle’s velocity within the plane normal to vector $\mathbf{B}$.)

8.3. Solve the dipole antenna radiation problem discussed in Sec. 2 (see Fig. 3) for the optimal length $l = \lambda/2$, assuming$^{43}$ that the current distribution in each of its arms is sinusoidal:

$$I(z,t) = I_0 \cos \frac{\pi z}{l} \cos \omega t.$$
8.4. Use the harmonic oscillator model of a bound charge, given by Eq. (7.30), to explore the transition between two scattering limits discussed in Sec. 3, in particular the resonant scattering taking place at \( \omega \approx \omega_0 \). In this context, discuss the contribution of scattering into oscillator’s damping.

8.5. A sphere of radius \( R \), made of a material with constant permanent electric polarization \( P_0 \) and mass density \( \rho \), is free to rotate about its center of mass. Calculate the total cross-section of scattering, by the sphere, of a linearly polarized electromagnetic wave of frequency \( \omega \ll R/c \), propagating in free space, in the limit of small wave amplitude, assuming that the initial orientation of the polarization vector \( \mathbf{P}_0 \) is random.

8.6. Use the Born approximation to analyze the interference pattern produced by plane wave’s scattering on a set of \( N \) similar, equidistant points on a straight line normal to the direction of the incident wave’s propagation – see Fig. on the right. Discuss the trend(s) of the pattern in the limit \( N \to \infty \).

8.7. Use the Born approximation to calculate the differential cross-section of plane wave scattering by a dielectric cube of side \( a \), with \( \varepsilon \approx \varepsilon_0 \). In the limits \( ka \ll 1 \) and \( ka \gg 1 \) (where \( k \) is the wave vector), analyze the angular dependence of the differential cross-section. Calculate the full cross-section for the simplest case when the incident wave vector is parallel to one of cube’s sides.

8.8. Use the Born approximation to calculate the differential cross-section of plane wave scattering by a nonmagnetic, uniform dielectric sphere with \( \varepsilon \approx \varepsilon_0 \), of an arbitrary radius \( R \). In the limits \( kR \ll 1 \) and \( 1 \ll kR \) (where \( k \) is the wave number), analyze the angular dependence of the differential cross-section, and calculate the full cross-section.

8.9. A sphere of radius \( R \) is made of a uniform, nonmagnetic, linear dielectric material. Calculate its full cross-section of scattering of a low-frequency monochromatic wave, with \( k \ll 1/R \), for an arbitrary dielectric constant, and compare the result with the solution of the previous problem.

8.10. Solve the previous problem, also in the low-frequency limit \( kR \ll 1 \), for the case when the sphere’s material has a frequency-independent Ohmic conductivity, and \( \varepsilon_{\text{opt}} = \varepsilon_0 \), and a relatively large skin depth (\( \delta >> R \)), and compare the results.

8.11. Use the Born approximation to calculate the differential cross-section of plane wave scattering on a right, circular cylinder of length \( l \) and radius \( R \), for arbitrary incidence.

8.12. Formulate the quantitative condition of the Born approximation validity for a uniform linear-dielectric scatterer with all linear dimensions of the order of \( a \).

8.13. Use the Huygens principle to calculate wave’s intensity on the symmetry plane of the slit diffraction experiment (i.e. at \( x = 0 \) in Fig. 11), for arbitrary ratio \( z/ka^2 \).
8.14. A plane wave with wavelength $\lambda$ is normally incident on an opaque, plane screen, with a round orifice of radius $R \gg \lambda$. Use the Huygens principle to calculate passed wave’s intensity distribution along system’s symmetry axis, at distances $z \gg R$ from the screen (see Fig. on the right), and analyze the result.

8.15. A plane monochromatic wave is normally incident on an opaque circular disk of radius $R \gg \lambda$. Use the Huygens principle to calculate wave’s intensity at distance $z \gg R$ behind the disk center (see Fig. on the right). Discuss the result.

8.16. Use the Huygens principle to analyze the Fraunhofer diffraction of a plane wave normally incident on a square-shape hole, of size $a \times a$, in an opaque screen. Sketch the diffraction pattern you would observe at a sufficiently large distance, and quantify expression “sufficiently large” for this case.

8.17. Within the Fraunhofer approximation, analyze the pattern produced by a 1D diffraction grating with the periodic transparency profile shown in Fig. on the right, for the normal incidence of a plane, monochromatic wave.

8.18. $N$ equal point charges are attached, at equal intervals, to a circle rotating with a constant angular velocity about its center – see Fig. on the right. For what values of $N$ does the system emit:
   (i) the electric dipole radiation?
   (ii) the magnetic dipole radiation?
   (iii) the electric quadrupole radiation?

8.19. The orientation of a magnetic dipole $\mathbf{m}$, of a fixed magnitude, is rotating about a certain axis with angular velocity $\omega$, with angle $\theta$ between them staying constant. Calculate the angular distribution and the average power of its radiation (into free space).

8.20. Complete the solution of the problem started in Sec. 9, by calculating the full power of radiation of the system of two charges oscillating in antiphase along the same straight line - see Fig. 6. Also, calculate the average radiation power for the case of harmonic oscillations, $d(t) = a\cos\omega t$, compare it with the case of a single charge performing similar oscillations, and interpret the difference.
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