Chapter 4. DC Currents

In this chapter I discuss the laws governing the distribution of constant ("dc") currents inside conducting media, with a focus on the linear ("Ohmic") conductivity. In most cases, the partial differential equation governing the distribution may be reduced to the same Laplace and Poisson equations whose solution methods have been discussed in detail in Chapter 2. Due to this fact, this chapter is rather short.

4.1. Continuity equation and the Kirchhoff laws

Until this point, our discussion of conductors has been limited to the cases when they are separated with insulators (meaning either vacuum or dielectric media) preventing any continuous motion of charges from one conductor to another, even if there is a voltage difference (and hence electric field) between them – see Fig. 1a.

Now let us connect two conductors galvanically, say with a wire – a thin, elongated conductor (Fig. 1b). Then the electric field causes the motion of charges in the wire - from a conductor with a higher electrostatic potential toward that with a lower potential, until the potentials equilibrate. Such process is called charge relaxation. The main equation governing this process may be obtained from the experimental fact (already mentioned in Sec. 1.1) that electric charges cannot appear or disappear (though opposite charges may recombine with the conservation of the net charge.) As a result the change of charge $Q$ in one conductor may change only due to the current $I$ through the wire:¹

$$\frac{dQ}{dt} = -I. \quad (4.1)$$

¹ Just as a (hopefully, unnecessary :-) reminder, in the SI units the current is measured in amperes (A). In the legal metrology, the ampere (rather than the coulomb, which is defined as $1C = 1A \times 1s$) is a primary unit. I will mention its formal definition in the next chapter. In the Gaussian units, Eq. (1) remains the same, so that the current’s unit is the so-called statampere - defined as statcoulomb per second.

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Let us express this law in a differential form, introducing the notion of *current density* vector \( \mathbf{j}(r) \). This vector may be defined via the following relation for current \( dI \) crossing an elementary area \( dA \) (Fig. 2)

\[
dI = j dA \cos \theta = (j \cos \theta) dA = j_x dA,
\]

where \( \theta \) is the angle between the normal to the surface and the carrier motion direction (which is taken for the direction of vector \( \mathbf{j} \)).

![Fig. 4.2. Current density vector.](image)

With that definition, Eq. (1) may be re-written as

\[
\frac{d}{dt} \int_V \rho d^3r = -\oint_S j_x d^2r,
\]

where \( V \) is an arbitrary stationary volume limited by the closed surface \( S \). Applying to this volume the same divergence theorem as was repeatedly used in previous chapters, we get

\[
\int_V \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} \right] d^3r = 0.
\]

Since volume \( V \) if arbitrary, this equation may be true only if

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0.
\]

This is the fundamental *continuity equation* - which is true even for the time-dependent phenomena.\(^2\)

The charge relaxation is of course a dynamic, time-dependent process. However, electric currents may also exist in stationary situations, when a current source, for example a *battery*, replenishes the conductor charges and hence sustains currents at a certain time-independent level – see Fig. 1c. (As we will see below, in most cases this process requires a persistent replenishment of the electrostatic energy from either a source or storage of energy of a different kind – say, the chemical energy of the battery.) Let us discuss the laws governing the distribution of such *dc currents*. In this case \( (\partial/\partial t = 0) \), Eq. (5) reduces to a very simple equation

\[
\nabla \cdot \mathbf{j} = 0.
\]

This equation acquires an even a simpler form in the particular but important case of *electric circuits* (Fig. 3), the systems may be presented as an electric connection of components of two types:

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\(^2\) Similar differential relations are valid for the density of any conserved quantity, for example for mass in classical fluid dynamics (see, e.g., CM Sec. 8.2), and for the probability in statistical physics (SM Sec. 5.6) and quantum mechanics (QM Sec. 1.4).
(i) small-size (*lumped*) *circuit elements* (also called “two-terminal devices”), meaning a passive resistor, a current source, etc. – generally, any black box with two wires sticking out of it, and

(ii) *perfectly conducting wires*, with negligible voltage drop along them, that are galvanically connected at certain points, called *nodes* (or “junctions”).

In the standard circuit theory, the electric charges of the nodes are considered negligible, and we may integrate Eq. (6) over the closed surface drawn around any node to get

$$\sum_j I_j = 0, \quad (4.7a)$$

where the summation is over all the wires (numbered with index \(j\)) connected in the node. On the other hand, according to its definition (2.25), voltage drop \(V_k\) across each circuit element may be presented as the difference of potentials of the adjacent nodes, \(V_k = \phi_k - \phi_{k-1}\). Summing such differences around any closed loop of the circuit (Fig. 3), we get all terms cancelled, so that

$$\sum_k V_k = 0. \quad (4.7b)$$

These relations are called, respectively, the 1st and 2nd *Kirchhoff laws* - or sometimes the *node rule* and the *loop rule*. They may seem elementary, and the genuine power of the Kirchhoff approach is in the fact a set of Eqs. (7), covering every node and every circuit element of the system, gives a system of equations sufficient for the calculation of all currents and voltages in it - provided that the relation between current and voltage in known for each circuit element.

It is almost evident that in the absence of current sources, the system of equations (7) has only a trivial solution: \(I_j = 0, V_k = 0\) - with the exotic exception of superconductivity, to be discussed in Sec. 6.3. The current sources, that allow non-vanishing current flows, may be described by their *electromotive forces* (*e.m.f.* \(V_k\), having the dimensionality of voltage, which have to be taken into account in the corresponding terms \(V_k\) of sum (7b). Let me hope that the reader has some experience of using Eqs. (7) for the analysis of simple circuits – say consisting of several resistors and dc batteries – so I may save time on a discussion of these simple problems.

4.2. The Ohm law

As was mentioned above, the relations spelled out in Sec. 1 are sufficient for forming a closed system of equations for finding currents and electric field in a system only if they are complemented
with material equations relating scalars \( I \) and \( V \) in each circuit element, i.e. vectors \( \mathbf{j} \) and \( \mathbf{E} \) in each point of the medium of such an element. The simplest, and most frequently met relation of this kind is the famous Ohm law whose differential form is

\[
\mathbf{j} = \sigma \mathbf{E},
\]

(4.8)

where \( \sigma \) is a constant called conductivity.\(^3\) Though this is not a fundamental relation, and is approximate for any conducting media, we can argue that if:

(i) there is no current at \( \mathbf{E} = 0 \) (mind superconductors!),
(ii) the medium is isotropic or almost isotropic (a notable exception: some organic conductors),
(iii) the mean free path \( l \) of current carriers is much smaller than the characteristic scale \( a \) of the spatial variations of \( \mathbf{j} \) and \( \mathbf{E} \),

then the Ohm law may be viewed as a result of the Taylor expansion of the local relation \( \mathbf{j}(\mathbf{E}) \) in relatively small fields, and thus is very common.

Table 1 gives the experimental values of dc conductivity for some practically important (or just representative) materials. The reader can see that the range of its values is very broad, covering more that 30 orders of magnitude, even without going to such extremes as very pure metallic crystals at very low temperatures, where \( \sigma \) may reach \( \sim 10^{12} \) S/m.

### Table 4.1. Ohmic conductivities for some representative (or practically important) materials at 20°C.

<table>
<thead>
<tr>
<th>Material</th>
<th>( \sigma ) (S/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Teflon ([C(_2)F(_4)](_n))</td>
<td>(10^{-22}-10^{-24})</td>
</tr>
<tr>
<td>Silicon dioxide</td>
<td>(10^{-16}-10^{-19})</td>
</tr>
<tr>
<td>Various glasses</td>
<td>(10^{-10}-10^{-14})</td>
</tr>
<tr>
<td>Deionized water</td>
<td>(\sim 10^{-6})</td>
</tr>
<tr>
<td>Sea water</td>
<td>5</td>
</tr>
<tr>
<td>Silicon ( n )-doped to ( 10^{16} ) cm(^{-1})</td>
<td>(2.5 \times 10^2)</td>
</tr>
<tr>
<td>Silicon ( n )-doped to ( 10^{19} ) cm(^{-1})</td>
<td>(1.6 \times 10^4)</td>
</tr>
<tr>
<td>Silicon ( p )-doped to ( 10^{19} ) cm(^{-1})</td>
<td>(1.1 \times 10^4)</td>
</tr>
<tr>
<td>Nichrome (alloy 80% Ni + 20% Cr)</td>
<td>(0.9 \times 10^6)</td>
</tr>
<tr>
<td>Aluminum</td>
<td>(3.8 \times 10^7)</td>
</tr>
<tr>
<td>Copper</td>
<td>(6.0 \times 10^7)</td>
</tr>
<tr>
<td>Zinc crystal along ( a )-axis</td>
<td>(1.65 \times 10^7)</td>
</tr>
<tr>
<td>Zinc crystal along ( c )-axis</td>
<td>(1.72 \times 10^7)</td>
</tr>
</tbody>
</table>

\(^3\) In SI units, the conductivity is measured in siemens per meter, where one siemens (S) is the reciprocal of one ohm: \( 1 \) S \( \equiv (1 \) \( \Omega \)\)\(^3\) \( \equiv 1 \) A / 1 V. The constant reciprocal to conductivity, \( 1/\sigma \), is called resistivity, and is commonly denoted by letter \( \rho \). I will, however, try to avoid using this notion, because I am already overusing this letter.
In order to get some feeling what do these values mean, let us consider a very simple system (Fig. 4): a plane capacitor of area $A \gg d^2$, filled with a material that has not only a dielectric constant $\varepsilon_r$, but also some Ohmic conductivity $\sigma$, with much more conductive plate electrodes.

Assuming that these properties are compatible with each other, we may assume that the distribution of electric potential (not too close to the capacitor edges) still obeys Eq. (2.39), so that the electric field is vertical and uniform, with $E = V/d$. Then, according to Eq. (6) the current density is also uniform, $j = \sigma E = \sigma V/d$. From here, the total current between the plates is

$$I = jA = \sigma EA = \sigma \frac{V}{d} A. \quad (4.9)$$

On the other hand, from Eqs. (2.26) and (3.45), the instant value of plate charge is $Q = C_m V = (\varepsilon_r \varepsilon_0 A/d) V$. Plugging these relations into Eq. (1), we see that the speed of charge (and voltage) relaxation does not depend on the geometric parameters $A$ and $d$:

$$\frac{dV}{dt} = -\frac{V}{\tau_r}, \quad \tau_r \equiv \frac{\varepsilon_r \varepsilon_0}{\sigma}, \quad (4.10)$$

where parameter $\tau_r$ has the sense of the relaxation time constant. As we know (see Table 3.1), for most practical materials the dielectric constant is within one order of magnitude from 10, so that the nominator of Eq. (10) is of the order of $10^{-10}$. As a result, according to Table 1, the charge relaxation time ranges from $\sim 10^{14}$s (more than a million years!) for best insulators like teflon, to $\sim 10^{-18}$s for the least resistive metals.

What is the physics behind these values of $\sigma$ and why, for some materials, Table 1 gives them with such a large uncertainty? If charge carriers move as classical particles (e.g., in plasmas or non-degenerate semiconductors), a reasonable description of conductivity is given by the famous Drude formula. In his picture, due to weak electric field, the charge carriers are accelerated in its direction (possibly on the top of their random motion in all directions, i.e. with a vanishing average velocity vector):

$$\frac{dv}{dt} = \frac{q}{m} E, \quad (4.11)$$

and as a result their velocity acquires an the average value

$$\langle v \rangle = \frac{dv}{dt} \tau = \frac{q}{m} E \tau, \quad (4.12)$$

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4 As will be discussed in Chapter 6, such simple analysis is only valid if $\sigma$ is not too high.
5 It was suggested by P. Drude in 1900.
where the phenomenological parameter $\tau = l/v$ (not to be confused with $\tau$!) may be understood as the effective average time between carrier scattering events. From here, the current density:

$$j = qn\langle v \rangle = \frac{q^2 n \tau}{m} E, \quad \text{i.e. } \sigma = \frac{q^2 \tau}{m}.$$  \hspace{1cm} (4.13a)

(Notice the independence of $\sigma$ of the carrier charge sign.) Another form of the same result, more popular in the physics of semiconductors, is

$$\sigma = q^2 \eta \mu, \quad \text{with } \mu = \frac{\tau}{m}. \hspace{1cm} (4.13b)$$

where parameter $\mu$, defined by relation $\langle v \rangle \equiv \mu E$, is called the charge carrier mobility.

Most good conductors (e.g., metals) are essentially degenerate Fermi gases (or liquids), in which the average thermal energy of a particle, $k_B T$ is much lower than the Fermi energy $\varepsilon_F$. In this case, a quantum theory is needed for the calculation of $\sigma$. Such theory was developed by the quantum physics’ godfather A. Sommerfeld in 1927 (and is sometimes called the Drude-Sommerfeld model). I have no time to discuss it in this course,\(^6\) and here I will only notice that for an ideal, isotropic Fermi gas the result is reduced to Eq. (13), with a certain effective value of $\tau$, so it may be used for estimates of $\sigma$, with due respect to the quantum theory of scattering. In a typical metal, $n$ is very high ($\sim 10^{23}$ cm$^{-3}$) and is fixed by the atomic structure, so that the sample quality may only affect $\sigma$ via the scattering time $\tau$.

At room temperature, the scattering of electrons by thermally-excited lattice vibrations (phonons) dominates, so that $\tau$ and $\sigma$ are high but finite, and do not change much from one sample to another. (Hence, the more accurate values given for metals in Table 1.) On the other hand, at $T \to 0$, a perfect crystal should not exhibit scattering at all, and conductivity should be infinite. In practice, this is never true (for example, due to electron scattering from imperfect boundaries of finite-size samples), and the effective conductivity $\sigma$ is infinite (or practically infinite, at least above the measurable value $\sim 10^{20}$ S/m) only in superconductors.\(^7\)

On the other hand, the conductivity of quasi-insulators (including deionized water) and semiconductors depends mostly of the carrier density $n$ that is much lower than in metals. From the point of view of quantum mechanics, this happens because the ground-state eigenenergies of charge carriers are localized within an atom (or molecule), and separated from excited states, with space-extended wavefunctions, by a large energy gap (called bandgap). For example, in SiO$_2$ the bandgap approaches 9 eV, equivalent to $\sim 4,000$ K. This is why, even at room temperatures the density of thermally-excited free charge carriers in good insulators is negligible. In these materials, $n$ is determined by impurities and vacancies, and may depend on a particular chemical synthesis or other fabrication technology, rather than on fundamental properties of the material. (On the contrary, the carrier mobility $\mu$ in these materials is almost technology-independent.)

The practical importance of the technology may be illustrated on the following example. In cells of the so-called floating-gate memories, in particular the flash memories, which currently dominate the nonvolatile digital memory technology, data bits are stored as small electric charges ($Q \sim 10^{-16}$ C) of

\(^6\) For such a discussion see, e.g., SM Sec. 6.3.

\(^7\) Electrodynamic properties of superconductors are so interesting (and important) that I will discuss them in more detail in Chapter 6.
highly doped silicon islands (so-called *floating gates*) separated from the rest of the integrated circuit with a ~10-nm-thick layer of silicon dioxide, SiO$_2$. Such layers are fabricated by high-temperature oxidation of virtually perfect silicon crystals. The conductivity of the resulting high-quality (though amorphous) material is so low, $\sigma \sim 10^{-19}$ S/m, that the relaxation time $\tau_r$, defined by Eq. (10), is well above 10 years – the industrial standard for data retention in non-volatile memories. In order to appreciate how good this technology is, the cited value should be compared with the typical conductivity $\sigma \sim 10^{-16}$ S/m of the usual, bulk SiO$_2$ ceramics.$^8$

4.3. Boundary problems

For an Ohmic conducting media, we may combine Eqs. (6) and (8) the following differential equation

$$\nabla \cdot (\sigma \nabla \phi) = 0 .$$

(4.14)

For a uniform conductor ($\sigma =$ const), Eq. (14) is reduced to the Laplace equation for the electrostatic potential $\phi$. As we already know from Chapters 2 and 3, its solution depends on the boundary conditions. These conditions depend on the interface type.

(i) Conductor-conductor interface. Applying the continuity equation (6) to a Gauss-type pillbox at the interface of two different conductors (Fig. 5), we get

$$(j_n)_1 = (j_n)_2 ,$$

(4.15)

so that if the Ohm law is valid inside each medium, then

$$\sigma_1 \frac{\partial \phi_1}{\partial n} = \sigma_2 \frac{\partial \phi_2}{\partial n} .$$

(4.16)

\[\text{Fig. 4.5. DC current “refraction” at the interface between two different conductors.}\]

Also, since the electric field should be finite, its potential $\phi$ has to be continuous across the interface - the condition that may also be written as

---

$^8$ Unfortunately, these notes are not an appropriate platform to discuss details of the floating-gate memory technology. However, I think that every educated physicist should know its basics, because such memories are presently the driver of all semiconductor integrated circuit technology development, and hence of the whole information technology progress. Perhaps the best available book is J. Brewer and M. Gill (eds.), *Nonvolatile Memory Technologies with Emphasis on Flash*, IEEE, 2008.
Both these conditions (and hence the solutions of the boundary problems using them) are similar to those for the interface between two dielectrics – cf. Eqs. (3.46)-(3.47).

Note that using the Ohm law, Eq. (17) may be rewritten as

\[
\frac{1}{\sigma_1}(j_\tau)_1 = \frac{1}{\sigma_2}(j_\tau)_2.
\]

Comparing it with Eq. (15) we see that, generally, the current density magnitude changes at the interface: \( j_1 \neq j_2 \). It is also curious that if \( \sigma_1 \neq \sigma_2 \), the current line slope changes at the interface (Fig. 4), qualitatively to the refraction of light rays in optics – see Chapter 7.

(ii) Conductor-electrode interface. The definition of an electrode, or a “perfect conductor”, is a medium with \( \sigma \to \infty \). Then, at fixed current density at the interface, the electric field in the electrode tends to zero, and hence it may be described by equation

\[
\phi = \phi_j = \text{const},
\]

where constants \( \phi_j \) may be different for different electrodes (numbered with index \( j \)). Note that with such boundary conditions the Laplace boundary problem becomes exactly the same as in electrostatics – see Eq. (2.35) – and hence we can use all the methods (and some solutions :-) of Chapter 2 for finding dc current distribution.

(iii) Conductor-insulator interface. For the description of an insulator, we can use \( \sigma = 0 \), so that Eq. (16) yields the following boundary condition,

\[
\frac{\partial \phi}{\partial n} = 0,
\]

for the potential derivative inside the conductor. From the Ohm law we see that this is just the very natural requirement for the dc current not to flow into an insulator.

Now, note that this condition makes the Laplace problem inside the conductor completely well-defined, and independent on the potential distribution in the adjacent insulator. On the contrary, due to the continuity of the electrostatic potential at the border, its distribution in the insulator has to follow that inside the conductor. Let us discuss this conceptual issue on the following (apparently, trivial) example: dc current in a long wire with a constant cross-section area \( A \). The reader certainly knows the answer:

\[
I = \frac{V}{R}, \quad \text{where} \quad R \equiv \frac{V}{I} = \frac{l}{\sigma A},
\]

where \( l \) is the wire length, and constant \( R \) is called the resistance.\(^9\) However, let us get this result formally from our theoretical framework. For the ideal geometry shown in Fig. 6a, this is easy to do. Here the potential evidently has a linear 1D distribution

\[^9\] The first of Eqs. (21) is essentially the integral form of the Ohm law (8), and is valid not only for a uniform wire, but for any Ohmic conductor with a geometry in which \( I \) and \( V \) may be clearly defined.
both in the conductor and the surrounding free space, with both boundary conditions (16) and (17) satisfied at the conductor-insulator interfaces, and condition (20) satisfied at the conductor-electrode interfaces. As a result, the electric field is constant and has only one component \( E_z = V/l \), so that inside the conductor

\[
\begin{align*}
    j_z &= \sigma E_z, \quad I = j_z A, \\
    \phi &= \text{const} - \frac{x}{l} V,
\end{align*}
\]

(4.22)

(4.23)

(4.24)

In order to find coefficient \( b_1 \), we have to use the boundary condition (20) at \( r = R \):

However, what about the geometry shown in Fig. 6b? In this case the field distribution in the insulator is dramatically different, but according to boundary problem defined by Eqs. (14) and (20), inside the conductor the solution is exactly the same as it was in the former case. Now, the Laplace equation in the surrounding insulator has to be solved with the boundary values of the electrostatic potential, “dictated” by the distribution of the current (and hence potential) in the conductor.

Let us solve a problem in that this conduction hierarchy may be followed analytically to the very end. Consider an empty spherical cavity cut in a conductor with an initially uniform current flow with constant density \( j_0 = n_0 \) (Fig. 7a). Following the conduction hierarchy, we have to solve the boundary problem in the conducting part of the system, i.e. outside the sphere \( (r \geq R) \), first. Since the problem is evidently axially-symmetric, we already know the general solution of the Laplace equation – see Eq. (2.172). Moreover, we know that in order to match the uniform field at \( r \to \infty \), all coefficients \( a_l \) but one \( (a_1 = -E_0 = -j_0/\sigma) \) have to be zero, and that the boundary conditions at \( r = R \) will give zero solutions for all coefficients \( b_l \) but one \( (b_1) \), so that

\[
\phi = -\frac{j_0}{\sigma}r \cos \theta + \frac{b_1}{r^2} \cos \theta, \quad \text{for } r \geq R. 
\]
\[ \frac{\partial \phi}{\partial r} \bigg|_{r=R} = \left( -\frac{j_0}{\sigma} - \frac{2h}{R^3} \right) \cos \theta = 0. \]  

(4.25)

This gives \( b_1 = -j_0 R^3/2\sigma \), so that, finally,

\[ \phi(r, \theta) = -\frac{j_0}{\sigma} \left( r + \frac{R^3}{2r^3} \right) \cos \theta. \]  

(4.26)

(Note that this potential distribution corresponds to the dipole moment \( p = -E_0 R^3/2 \). It is easy to check that if the empty sphere was cut in a dielectric, the potential distribution outside the cavity would be similar, with \( \mathbf{p} = -E_0 R^3 (\varepsilon_r - 1)/(\varepsilon_r + 2) \). In the limit \( \varepsilon_r \to \infty \), these two results coincide, despite the rather different type of the problem: in the dielectric case, there is no current at all.)

Now, as the second step in the conductivity hierarchy, we may find the electrostatic potential distribution \( \phi(r, \theta) \) in the insulator, in this particular case inside the cavity \( (r \leq R) \). It should also satisfy the Laplace equation with the boundary conditions at \( r = R \), “dictated” by distribution (26):

\[ \phi(R, \theta) = -\frac{3j_0}{2\sigma} R \cos \theta. \]  

(4.27)

We could again solve this problem by the formal variable separation (keeping in the general solution (2.172) only the term proportional to \( b_l \), that does not diverge at \( r \to 0 \), but if we notice that boundary condition (27) depends on just one Cartesian coordinate, \( z = R \cos \theta \), the solution may be just guessed:

\[ \phi(r, \theta) = -\frac{3j_0}{2\sigma} z = -\frac{3j_0}{2\sigma} r \cos \theta, \quad \text{at } r \leq R. \]  

(4.28)

It evidently satisfies the Laplace equation and the boundary condition (27), and corresponds to a constant vertical electric field equal to \( 3j_0/2\sigma \) - see Fig. 6b.

The conductivity hierarchy says that static electrical fields and charges outside conductors (e.g., electric wires) do not affect currents flowing in the wires, and it is physically clear why. For example, if
a charge in vacuum is slowly moved close to a wire, it (in accordance with the linear superposition principle) will only induce an additional surface charge (see Chapter 2) that screens the external charge’s field, without participating in (or disturbing) the current flow inside the conductor.

Besides the conceptual discussion, the two examples given above may be considered as a demonstration of the application of the first two methods described in Chapter 2 (the orthogonal coordinates (Fig. 5) and variable separation (Fig. 6)) to dc current distribution problems. Continuing this review of the methods we know, let us discuss the analog of the method of charge images. Let us consider the spherically-symmetric potential distribution of the electrostatic potential, similar to that given by Eq. (1.35):

$$\phi = \frac{c}{r}.$$  (4.29)

As we know from Chapter 1, this is a particular solution of the 3D Laplace equation at all points but \(r = 0\), and hence is a legitimate solution in a current-carrying conductor as well. In vacuum, this distribution would correspond to a point charge \(q = 4\pi\varepsilon_0 c\); but what about the conductor? Calculating the corresponding electric field and current density,

$$E = -\nabla\phi = \frac{c}{r^3} r, \quad j = \sigma E = \sigma \frac{c}{r^3} r,$$  (4.30)

we see that the total current flowing from the point in the origin through a sphere of an arbitrary radius \(r\) does not depend on the radius:

$$I = Aj = 4\pi r^2 j = 4\pi\sigma c.$$  (4.31)

Plugging the resulting \(c\) into Eq. (29), we get

$$\phi = \frac{I}{4\pi\sigma r}.$$  (4.32)

Hence the Coulomb-type distribution of the electric potential in a conductor is possible (at least at some distance from the singular point \(r = 0\)), and describes dc current \(I\) flowing out of a small-size electrode - or into such a point, if coefficient \(c\) is negative. Such current injection may be readily implemented experimentally; think for example about an insulated wire with a small bare end, inserted into a poorly conducting soil – an important method in geophysical research.\(^{10}\)

Now let the injection point \(r'\) be close to a plane interface between the conductor and an insulator (Fig. 8). In this case, besides the Laplace equation, we should satisfy the boundary condition,

$$j_n = \sigma E_n = -\sigma \frac{\partial\phi}{\partial n} = 0.$$  (4.33)

It is clear that this can be done by replacing the insulator for a conductor with an additional current injection point, at the mirror image point \(r''\). Note, however, that in contrast to the charge images, the sign of the imaginary current has to be \textit{similar}, not opposite, to the initial one, so that the total electrostatic potential inside the conducting semi-space is

\(^{10}\) Such situations are even more natural in 2D situations, for example, think about a wire soldered, in a small spot, to a thin metallic foil. (Note that here the current density distribution law is different, \(j \propto 1/r\) rather than \(1/r^2\).)
\[
\phi(r) = \frac{I}{4\pi\sigma} \left( \frac{1}{|r - r'|} + \frac{1}{|r - r''|} \right) \tag{4.34}
\]

(Note that the image current’s sign would be opposite if we discussed an interface between a conductor with a moderate conductivity and a perfect conductor (“electrode”) whose potential should be virtually constant.)

This result may be readily used, for example, to calculate the current density at the conductor’s surface, as a function of distance \(\rho\) from point 0 (the surface point closest to the current injection) – see Fig. 8. At the surface, Eq. (34) yields

\[
\phi = \frac{I}{2\pi\sigma} \frac{1}{\left(\rho^2 + d^2\right)^{3/2}}, \tag{4.35}
\]

so that the current density is independent of \(\sigma\):

\[
j_\rho = \sigma E_\rho = -\sigma \frac{\partial \phi}{\partial \rho} = \frac{I}{2\pi} \frac{\rho}{\left(\rho^2 + d^2\right)^{3/2}}. \tag{4.36}
\]

Deviations from Eqs. (35) and (36), which are valid for a uniform medium, may be used to find and characterize conductance inhomogeneities, say, those due to mineral deposits in the Earth crust.11

### 4.4. Dissipation power

Let me conclude this brief chapter with an ultra-short discussion of energy dissipation in conductors. In contrast to the electrostatics situations in insulators (vacuum or dielectrics), at dc conduction the electrostatic energy \(U\) is “dissipated” (i.e. transferred to heat) at a certain rate \(P \equiv -dU/dt\), called *dissipation power*.12 This rate may evaluated by calculating the power of electric field’s work on a single moving charge:

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11 In practice, the current injection may be produced, due to electrochemical reactions, by an ore mass itself, so that one need only measure (and interpret :-) the resulting potential distribution - the so-called *self-potential method* - see, e.g., Sec. 6.1 in monograph by W. Telford *et al.*, *Applied Geophysics*, 2nd ed., Cambridge U. Press, 1990.

12 Since the electric field and hence the electrostatic energy are time-independent, this means that the energy is replenished at the same rate from the current source(s).
\[ P_i = F \cdot v = qE \cdot v . \quad (4.37) \]

After the summation over all charges, Eq. (37) gives us the dissipation power. If the charge density \( n \) is uniform, multiplying by it both parts of this equation, and taking into account that \( qnv = j \), for the power dissipated in a unit volume we get the **Joule law**

\[
\rho \equiv \frac{P}{V} = \frac{P}{V} = \rho_{\text{N}} n = qE \cdot v n = E \cdot j . \quad (4.38)
\]

In the particular case of the Ohmic conductivity, this expression may be also rewritten in two other forms:

\[
\rho = \sigma E^2 = \frac{j^2}{\sigma} . \quad (4.39)
\]

At dc conduction, the energy is permanently replenished by a flow of power from the current source(s).

With our electrostatics background, it is straightforward (and hence left for reader’s exercise) to prove that the dc current distribution in a uniform Ohmic conductor, at a fixed voltage applied at its borders, corresponds to the minimum of the total dissipation power

\[
P = \sigma \int E^2 d^3 r . \quad (4.40)
\]

### 4.5. Exercise problems

#### 4.1. Find the resistance between two large conductors separated with a very thin, plane, insulating partition, with a circular hole of radius \( R \) in it – see Fig. on the right.

*Hint:* You may like to use the degenerate ellipsoidal coordinates that had been used in Sec. 2.4 to find the self-capacitance of a round disk in vacuum.

#### 4.2. Calculate the effective (average) conductivity \( \sigma_{\text{ef}} \) of a medium with many empty spherical cavities of radius \( R \), carved at random points in a uniform Ohmic conductor (see Fig. on the right), in the limit of low density \( n \ll R^{-3} \) of the spheres.

*Hint:* Try to use the analogy with a dipole media (Sec. 3.2).

#### 4.3. In two separate experiments, a narrow gap, of irregular width, between two close metallic electrodes is filled with some material - in the first case, with a uniform linear insulator with an electric permittivity \( \varepsilon \), and in the second case, with a uniform conducting material with an Ohmic conductivity \( \sigma \). Neglecting the fringe effects, calculate the relation between the mutual capacitance \( C \) between the electrodes (in the first case) and the dc resistance \( R \) between then (in the second case).
4.4. Calculate the voltage drop $V$ across a uniform, wide resistive slab of thickness $t$, at distance $l$ from the points of injection/ejection of dc current $I$ that is passed across the slab - see Fig. on the right.

*Hint*: Try to use the dc current analog of the charge image method.

4.5. Find the voltage drop $V$ between two corners of a square cut from a uniform, resistive sheet of thickness $t$, induced by dc current $I$ that is passed between its two other corners - see Fig. on the right.

4.6. Calculate the distribution of dc current density in a thin, round, uniform resistive disk, if the current is inserted into a point at its rim, and picked up at the center.

4.7. The simplest model of a vacuum diode consists of two plane, parallel metallic electrodes of area $A$, separated by a gap of thickness $d \ll A^{1/2}$: a “cathode” which emits electrons to vacuum, and an “anode” which absorbs the electrons arriving at its surface. Calculate the dc $I$-$V$ curve of the diode, i.e. the stationary relation between current $I$ flowing between the electrodes and voltage $V$ applied between them, using the following simplifying assumptions:

(i) due to the effect of the negative space charge of the emitted electrons, current $I$ is much smaller than the emission ability of the cathode,
(ii) the initial velocity of the emitted electrons is negligible, and
(iii) the direct Coulomb interaction of electrons (besides the space charge effect) is negligible.

4.8. Calculate the space-charge-limited current in a system with the same geometry, and using the same assumptions as in the previous problem, besides assuming now that the emitted charge carriers move not ballistically, but in accordance with the Ohm law, with the conductivity given by Eq. (4.13): $\sigma = q^2 \mu n$, with constant mobility $\mu$.

*Hint*: In order to get a realistic result, assume that the medium in which the carriers move has a certain dielectric constant $\varepsilon_r$.

4.9. Prove that the distribution of dc currents in a uniform Ohmic conductor, at fixed voltage applied at its boundaries, corresponds to the minimum of the total power dissipation (“Joule heat”).

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13 As was mentioned in Sec. 4.2 of the lecture notes, the assumption of constant (charge-density-independent) mobility is most suitable for semiconductors.