Chapter 1. Electric Charge Interaction

This brief chapter describes the basics of electrostatics, the study of interactions between static (or slowly moving) electric charges. Much of this material should be known to the reader from his or her undergraduate studies; because of that, the explanations will be very succinct.\(^1\)

1.1. The Coulomb law

A serious discussion of the Coulomb law\(^2\) requires a common agreement on the meaning of the following notions:\(^3\)

- electric charges \(q_k\), as revealed, most explicitly, by experimental observation of electrostatic interaction between the charged particles;

- electric charge conservation, meaning that the algebraic sum of \(q_k\) of all particles inside any closed volume is conserved, unless the charged particles cross the volume’s border; and

- a point charge, meaning the charge of an ultimately small (“point”) particle whose position in space may be completely described (in a given reference frame) by its radius-vector \(r = n_1 r_1 + n_2 r_2 + n_3 r_3\), where \(n_j\) (with \(j = 1, 2, 3\)) are unit vectors directed along 3 mutually perpendicular axes, and \(r_j\) are the corresponding Cartesian components of \(r\).

I will assume that these notions are well known to the reader - though my strong advice is to give some thought to their vital importance. Using them, the Coulomb law for the electrostatic interaction of two point charges in otherwise free space may be formulated as follows:

\[
F_{kk'} = \kappa q_k q_{k'} \frac{r_k - r_{k'}}{|r_k - r_{k'}|^3},
\]

where \(F_{kk'}\) denotes the force exerted on charge number \(k\) by charge number \(k'\). This law is certainly very familiar to the reader, but several remarks may still be due:

(i) Flipping indices \(k\) and \(k'\), we see that Eq. (1)\(^4\) complies with the 3\(^{rd}\) Newton law: the reciprocal force is equal in magnitude but opposite in direction: \(F_{k'k} = -F_{kk'}\).

(ii) According to Eq. (1), the magnitude of the force, \(F_{kk'}\), is inversely proportional to the square of the distance between the two charges – the well-known undergraduate-level formulation of the Coulomb law.

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\(^1\) For remedial reading, virtually any undergraduate text on electricity and magnetism may be used; I can recommend either the classical text by I. Tamm, *Fundamentals of Theory of Electricity*, Mir, 1979, or the more readily available textbook by D. Griffiths, *Introduction to Electrodynamics*, 3\(^{rd}\) ed., Prentice-Hall, 1999.

\(^2\) Discovered experimentally in the early 1780s, and formulated in 1785 by C.-A. de Coulomb.

\(^3\) On the top of the more general notions of classical Cartesian space, point particles and forces, which are used in classical mechanics – see, e.g., CM Sec. 1.1. (Acronyms CM, SM, and QM refer to other three parts of my lecture note series. In those parts, this Classical Electrodynamics part is referred to as EM.)

\(^4\) As in all other parts of my lecture notes, chapter numbers are omitted in references to equations, figures, and sections within the same chapter.
(iii) Since vector \((\mathbf{r}_k - \mathbf{r}_{k'})\) is directed from point \(\mathbf{r}_{k'}\) toward point \(\mathbf{r}_k\) (Fig. 1), Eq. (1) implies that charges of the same sign (i.e. with \(q_k q_{k'} > 0\)) repulse, while those with opposite signs (\(q_k q_{k'} < 0\)) attract each other.

(iv) Constant \(\kappa\) in Eq. (1) depends on the system of units we use. In the Gaussian units, \(\kappa\) is set to 1, for the price of introducing a special unit of charge (the statcoulomb) that would fit the experimental data for Eq. (1), for forces \(\mathbf{F}_{kk'}\) measured in Gaussian units (dynes). On the other hand, in the International System (“SI”) of units, the charge unit is one coulomb (abbreviated \(\text{C}\)), close to \(3 \times 10^9\) statcoulombs, and \(\kappa\) is different from unity: 

\[
\kappa_{\text{SI}} = \frac{1}{4\pi \varepsilon_0} \equiv 10^{-7} \text{c}^2. \tag{1.2}
\]

Fig. 1.1. Direction of the Coulomb forces (for \(q_k q_{k'} > 0\)).

Unfortunately, the continuing struggle between zealot proponents of these two systems bears all ugly features of a religious war, with a similarly slim chances for any side to win it in any foreseeable future. In my humble view, each of these systems has its advantages and handicaps (to be noted on several occasions below), and every educated physicist should have no problem with using any of them. Following insisting recommendations of international scientific unions, I will mostly use SI units, but for readers’ convenience, duplicate the most important formulas in the Gaussian units.

Besides Eq. (1), another key experimental law of electrostatics is the linear superposition principle: the electrostatic forces exerted on some point charge (say, \(q_k\)) by other charges do not affect each other and add up as vectors to form the net force:

\[
\mathbf{F}_k = \sum_{k' \neq k} \mathbf{F}_{kk'}, \tag{1.3}
\]

where the summation is extended over all charges but \(q_k\), and the partial force \(\mathbf{F}_{kk'}\) is described by Eq. (1). The fact that the sum is restricted to \(k' \neq k\) means that a point charge does not interact with itself.

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5 In the formal metrology, one coulomb is defined as the charge carried over by a constant current of one ampere (see Ch. 5 for its definition) during one second.

6 Constant \(\varepsilon_0\) is called either the electric constant or the free space permittivity; from Eq. (2) with the free-space speed of light \(c \approx 3 \times 10^8\) m/c, \(\varepsilon_0 \approx 8.85 \times 10^{-12}\) SI units. For more accurate values of the constants, and their brief discussion, see appendix CA: Selected Physical Constants.

7 Physically this is a very strong statement: it means that Eq. (1) is valid for any pair of charges regardless of presence of other charges, i.e. not only in the free space, but in also placed into an arbitrary medium. The apparent modification of this relation by conductors (Ch. 2) and dielectrics (Ch. 3) is just the result of appearance of additional electric charges within those media.
This fact may look trivial from Eq. (1), whose right-hand part diverges at \( r_k \rightarrow r_{k'} \), but becomes less evident (though still true) in quantum mechanics where the charge of even an elementary particle is effectively spread around some volume, together with particle’s wavefunction.\(^8\)

Now we may combine Eqs. (1) and (3) to get the following expression for the net force \( \mathbf{F} \) acting on some charge \( q \) located at point \( \mathbf{r} \):

\[
\mathbf{F} = q \frac{1}{4\pi\varepsilon_0} \sum_{r_k \neq \mathbf{r}} q_{k'} \frac{\mathbf{r} - \mathbf{r}_{k'}}{|\mathbf{r} - \mathbf{r}_{k'}|^3}.
\]

This equation implies that it makes sense to introduce the notion of the electric field at point \( \mathbf{r} \), as an entity independent of the probe charge \( q \), characterized by vector

\[
\mathbf{E}(\mathbf{r}) \equiv \frac{\mathbf{F}}{q},
\]

formally called the electric field strength – but much more frequently, just the “electric field”. In these terms, Eq. (4) becomes

\[
\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{r_k \neq \mathbf{r}} q_{k'} \frac{\mathbf{r} - \mathbf{r}_{k'}}{|\mathbf{r} - \mathbf{r}_{k'}|^3}.
\]

This concept is so appealing that Eq. (5) is used well beyond the boundaries of free-space electrostatics. Moreover, the notion of field becomes virtually unavoidable for description of time-dependent phenomena (such as electromagnetic waves), where the electromagnetic field shows up as a specific form of matter, with zero rest mass, and hence different from the usual “material” particles.

Many problems involve many point charges \( q_k, q_{k'}, \ldots \), located so closely that it is possible to approximate them with a continuous charge distribution. Indeed, for a group of charges within a very small volume \( d^3r' \), with the linear size satisfying strong condition \( dr \ll |\mathbf{r}_k - \mathbf{r}_{k'}| \), the geometrical factor in Eq. (6) is essentially the same. As a result, all these charges may be treated as a single charge \( d\mathcal{Q}(\mathbf{r}') \). Since this charge is proportional to \( d^3r' \), we can define the local (3D) charge density \( \rho(\mathbf{r}') \) by relation\(^9\)

\[
\rho(\mathbf{r}')d^3r' \equiv d\mathcal{Q}(\mathbf{r}') \equiv \sum_{r_k \in d^3r'} q_{k'},
\]

and rewrite Eq. (6) as

\[
\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{d^3r'} d\mathcal{Q}(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{1}{4\pi\varepsilon_0} \sum_{d^3r'} \rho(\mathbf{r}')d^3r' \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3},
\]

\(^8\) Moreover, there are some widely used approximations, e.g., the Kohn-Sham equations in the density functional theory of multiparticle systems, which essentially violate this law, thus limiting the accuracy and applicability of these approximations - see, e.g., QM Sec. 8.4.

\(^9\) The 2D (areal) charge density \( \sigma \) and 1D (linear) density \( \lambda \) may be defined absolutely similarly: \( d\mathcal{Q} = \sigma d\ell, d\mathcal{Q} = \lambda dr \). Note that a finite value of \( \sigma \) and \( \lambda \) means that the volume density \( \rho \) is infinite in the charge location points; for example for a plane \( z = 0 \), charged with a constant areal density \( \sigma, \rho = \sigma\delta(z) \).
i.e. as the integral (over the whole volume containing all essential charges):\(^{10}\)

\[
E(r) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(r') \frac{r-r'}{|r-r'|^3}}{d^3 r'}. \tag{1.9}
\]

It is very convenient that Eq. (9) may be used even in the case of discrete point charges, employing the notion of Dirac’s \(\delta\)-function,\(^{11}\) which is a mathematical approximation for a very sharp function equal to zero everywhere but one point, and still having a finite (unit) integral. Indeed, in this formalism, a set of point charges \(q_{k'}\) located in points \(r_{k'}\) may be presented by the pseudo-continuous distribution with density

\[
\rho(r') = \sum_k q_k \delta(r' - r_k). \tag{1.10}
\]

Plugging this expression into Eq. (9), we come back to the discrete version (6) of the Coulomb law.

1.2. The Gauss law

Due to the extension to point (“discrete”) charges, it may seem that Eqs. (5) and (9) is all we need for solving any problem of electrostatics. In practice, this is not quite true, first of all because the direct use of Eq. (9) frequently leads to complex calculations. Indeed, let us consider a very simple example: the electric field produced by a spherically-symmetric charge distribution with density \(\rho(r')\). We may immediately use the problem symmetry to argue that the electric field should be also spherically-symmetric, with only one component in spherical coordinates: \(E(r) = E(r)\mathbf{n}_r\), where \(\mathbf{n}_r \equiv r/r\) is the unit vector in the direction of the field observation point \(r\) (Fig. 2).

![Fig. 1.2. One of the simplest problems of electrostatics: electric field produced by a spherically-symmetric charge distribution.](image)

Taking this direction as the polar axis of a spherical coordinate system, we can use the evident independence of the elementary radial field \(dE\), created by the elementary charge \(\rho(r')d^3 r' = \rho(r')r^2\sin\theta dr'd\theta'd\phi'\), of the azimuth angle \(\phi'\), and reduce integral (9) to

\[
E = \frac{1}{4\pi\varepsilon_0} \int_0^{\pi} \sin\theta d\theta \int_0^\infty r^2 dr' \frac{\rho(r')}{(r'n)^2} \cos\theta, \tag{1.11}
\]

\(^{10}\) Note that for a continuous, smooth charge distribution, integral (9) does not diverge at \(R \equiv r - r' \rightarrow 0\), because in this limit the fraction under the integral increases as \(R^2\), i.e. slower than the decrease of the elementary volume \(d^3 r'\), proportional to \(R^3\).

\(^{11}\) See, e.g., Sec. 14 of the *Selected Mathematical Formulas* appendix, referred below as MA.
where $\theta$ and $r''$ are the geometrical parameters marked in Fig. 2. Since they all may be readily expressed via $r'$ and $\theta'$ using auxiliary parameters $a$ and $h$,

$$
cos \theta = \frac{r - a}{r''}, \quad (r'')^2 = h^2 + (r - r' \cos \theta)^2, \quad a = r' \cos \theta', \quad h = r' \sin \theta',
$$

integral (11) may be eventually reduced to an explicit integral over $r'$ and $\theta'$. and worked out analytically, but that would require some effort.

For more complex problem, integral (8) may be much more complex, defying an analytical solution. One could argue that with the present-day abundance of computers and numerical algorithm libraries, one can always resort to numerical integration. This argument may be enhanced by the fact that numerical integration is based on the replacement of the integral by a sum, and summation is much more robust to (unavoidable) discretization and rounding errors than the finite-difference schemes typical for the numerical solution of differential equations.

These arguments, however, are only partly justified, since in many cases the numerical approach runs into a problem sometimes called the curse of dimensionality, in which the last word refers to the number of input parameters of the problem to be solved, i.e. the dimensionality of its parameter space. Let us discuss this issue, because it is common for most fields of physics and, more generally, any quantitative science.\(^{12}\)

If the number of the parameters of a problem is small, the results of its numerical solution may be of the same (and in some sense higher) value than the analytical ones. For example, if a problem has no parameters, and its result is just one number (say, $\pi^2/4$), this “analytical” answer hardly carries more information than its numerical form 2.4674011… Now, if a problem has one input parameter (say, $a$), the result of an analytical approach in most cases may be presented as an analytical function $f(a)$. If the function is really simple, called elementary, with many properties well known (say, $f(a) = \sin a$), this function gives us virtually everything we want to know. However, if the function is complicated, you would need to tabulate it numerically for a set of values of parameter $a$ and possibly present the result as a plot. The same results (and the same plot) can be calculated numerically, without using analytics at all. This plot may certainly be very valuable, but since the analytical form has a potential of giving you more information (say, the values of $f(a)$ outside the plot range, or the asymptotic behavior of the function), it is hard to say that the numerics completely beat the analytics here.

Now let us assume that you have more input parameters. For two parameters (say, $a$ and $b$), instead of one curve you would need a family of such curves for several (sometimes many) values of $b$. Still, the plots sometimes may fit one page convenient for viewing, so it is still not too bad. Now, if you have three parameters, the full representation of the results may require many pages (maybe a book) full of curves, for four parameters we may speak about several bookshelves, for five parameters something like a library, etc. For large number of parameters, typical for many scientific problems, the number of points in the parameters space grows exponentially, even the volume of calculations necessary for the generation of this data may become impracticable, despite the dirt-cheap CPU time we have now.

Thus, despite the current proliferation of numerical methods in physics, analytical results have an ever-lasting value, and we should try to get them whenever we can. For our current problem of finding electric field generated by a fixed set of electric charges, large help comes from the Gauss law.

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\(^{12}\) Actually, the term “curse of dimensionality” was coined in the 1950s by R. Bellman in the context of the optimal control theory, and only later spread to other sciences that heavily rely on numerical calculations.
Let us consider a single point charge \( q \) inside a smooth, closed surface \( A \) (Fig. 3), and calculate product \( E_n d^2r \), where \( d^2r \) is an infinitesimal element of the surface (which may be well approximated with a plane of that area), and \( E_n \) is the component of the electric field in that point, normal to that plane.

This component may be calculated as \( E \cos \theta \), where \( \theta \) is the angle between vector \( \mathbf{E} \) and the unit vector \( \mathbf{n} \) normal to the surface. (Equivalently, \( E_n \) may be presented as the scalar product \( \mathbf{E} \cdot \mathbf{n} \).) Now let us notice that the product \( \cos \theta d^2r \) is nothing more than the area \( d^2r' \) of the projection of \( d^2r \) onto the plane perpendicular to vector \( \mathbf{r} \) connecting charge \( q \) with this point of the surface (Fig. 3), because the angle between the planes \( d^2r' \) and \( d^2r \) is also equal to \( \theta \). Using the Coulomb law for \( \mathbf{E} \), we get

\[
E_n d^2r = E \cos \theta d^2r = \frac{1}{4 \pi \varepsilon_0} \frac{q}{r^2} d^2r'.
\]  

(1.13)

But the ratio \( d^2r' / r^2 \) is nothing more than the elementary solid angle \( d\Omega \) under which the areas \( d^2r' \) and \( d^2r \) are seen from the charge point, so that \( E_n d^2r \) may be presented as just a product of \( d\Omega \) by a constant \((q / 4 \pi \varepsilon_0)\). Summing these products over the whole surface, we get

\[
\oint_S E_n d^2r = \frac{q}{4 \pi \varepsilon_0} \oint_S d\Omega = \frac{q}{\varepsilon_0},
\]  

(1.14)

since the full solid angle equals \( 4\pi \). (The integral in the left-hand part of this relation is called the flux of electric field through surface \( S \).)

Equation (14) expresses the Gauss law for one point charge. However, it is only valid if the charge is located inside the volume limited by the surface. In order to find the flux created by a charge outside of the volume, we still can use Eq. (13), but to proceed we have to be careful with the signs of the elementary contributions \( E_n dA \). Let us use the common convention to direct the unit vector \( \mathbf{n} \) out of the closed volume we are considering (the so-called outer normal), so that the elementary product \( E_n d^2r = (\mathbf{E} \cdot \mathbf{n})d^2r \) and hence \( d\Omega = E_n d^2r' / r^2 \) is positive if vector \( \mathbf{E} \) is pointing out of the volume (like in the example shown in Fig. 3a and the upper-right area in Fig. 3b), and negative in the opposite case (for example, in the lower-left area in Fig. 3b). As the latter figure shows, if the charge is located outside of the volume, for each positive contribution \( d\Omega \) there is always equal and opposite contribution to the
integral. As a result, at the integration over the solid angle the positive and negative contributions cancel exactly, so that
\[
\oint_S E_n d^2 r = 0. \tag{1.15}
\]

The real power of the Gauss law is revealed by its generalization to the case of many charges within volume \( V \). Since the calculation of flux is a linear operation, the linear superposition principle (3) means that the flux created by several charges is equal to the (algebraic) sum of individual fluxes from each charge, for which either Eq. (14) or Eq. (15) are valid, depending on the charge position (in or out of the volume). As the result, for the total flux we get:
\[
\oint_S E_n d^2 r = \frac{Q_V}{\varepsilon_0} \sum_{r_j \in V} q_j = \frac{1}{\varepsilon_0} \int_V \rho(r') d^3 r', \tag{1.16}
\]
where \( Q_V \) is the net charge inside volume \( V \). This is the full version of the Gauss law.

In order to appreciate the problem-solving power of the law, let us return to the problem presented in Fig. 2, i.e. a spherical charge distribution. Due to its symmetry, which had already been discussed above, if we apply Eq. (16) to a sphere of radius \( r \), the electric field should be perpendicular to the sphere at each its point (i.e., \( E_n = E \)), and its magnitude the same at all points: \( E_n = E = E(r) \). As a result, the flux calculation is elementary:
\[
\oint_S E_n d^2 r = 4\pi r^2 E(r). \tag{1.17}
\]

Now, applying the Gauss law (16), we get:
\[
4\pi r^2 E(r) = \frac{1}{\varepsilon_0} \int_{r<r} \rho(r') d^3 r' = \frac{4\pi}{\varepsilon_0} \int_0^r r'^2 \rho(r') dr', \tag{1.18}
\]
so that, finally,
\[
E(r) = \frac{1}{r^2 \varepsilon_0} \int_0^r r'^2 \rho(r') dr' = \frac{1}{4\pi \varepsilon_0} \frac{Q(r)}{r^2}, \tag{1.19}
\]
where \( Q(r) \) is the full charge inside the sphere of radius \( r \):
\[
Q(r) \equiv 4\pi \int_0^r \rho(r') r'^2 dr'. \tag{1.20}
\]

In particular, this formula shows that the field \textit{outside} of a sphere of a finite radius \( R \) is exactly the same as if all its charge \( Q = Q(R) \) was concentrated in the sphere’s center. (Note that this important result is only valid for any spherically-symmetric charge distribution.) For the field \textit{inside} the sphere, finding electric field still requires an explicit integration (20), but this 1D integral is much simpler than the 2D integral (11), and in some important cases may be readily worked out analytically. For example, if charge \( Q \) is uniformly distributed inside a sphere of radius \( R \),
\[
\rho(r') = \rho = \frac{Q}{V} = \frac{Q}{(4\pi/3)R^3}, \tag{1.21}
\]
the integration is elementary:

\[ E(r) = \frac{\rho}{r^2 \varepsilon_0} \int_0^r r'^2 \, dr' = \frac{\rho r}{3 \varepsilon_0} = \frac{1}{4 \pi \varepsilon_0} \frac{Q r}{R^3}. \]  

(1.22)

We see that in this case the field is growing linearly from the center to the sphere’s surface, and only at \( r > R \) starts to decrease in agreement with Eq. (19) with constant \( Q(r) = Q \). Another important observation is that the results for \( r \leq R \) and \( r \geq R \) give the same value \((Q/4 \pi \varepsilon_0 R^2)\) at the charged sphere’s surface, \( r = R \), so that the electric field is continuous.

In order to underline the importance of the last fact, let us consider one more elementary but very important example of the Gauss law’s application. Let a thin plane sheet (Fig. 4) be charged uniformly, with an areal density \( \sigma = \text{const} \) (see Footnote 9 above).

In this case, it is fruitful to use the Gauss volume in the form of a planar “pillbox” of thickness \( 2z \) (where \( z \) is the Cartesian coordinate perpendicular to charged plane) and certain area \( A \) – see Fig. 4. Due to the symmetry of the problem, it is evident that the electric field should be: (i) directed along axis \( z \), (ii) constant on each of the upper and bottom sides of the pillbox, (iii) equal and opposite on these sides, and (iv) parallel to the side surfaces of the box. As a result, the full electric field flux through the pillbox surface is just \( 2AE(z) \), so that the Gauss law (16) yields

\[ 2AE(z) = \frac{1}{\varepsilon_0} Q_A = \frac{1}{\varepsilon_0} \sigma A, \]

(1.23)

and we get a very simple but important formula

\[ E(z) = \frac{\sigma}{2\varepsilon_0} = \text{const}. \]  

(1.24)

Notice that, somewhat counter-intuitively, the field magnitude does not depend on the distance from the charged plane. From the point of view of the Coulomb law (5), this result may be explained as follows, the farther the observation point from the plane, the weaker the effect of each elementary charge, \( dQ = \sigma d^2 r \), but the more such elementary charges give contributions to the vertical component of vector \( E \).

Note also that though the magnitude \( E \equiv |E| \) of the electric field is constant, its vertical component \( E_z \) changes sign at \( z = 0 \) (Fig. 4), experiencing a discontinuity (jump) equal to \( \Delta E_z = \sigma \varepsilon_0 \). This jump disappears if the surface is not charged \((\sigma = 0)\). This statement remains true in a more general case of finite volume (but not surface!) charge density \( \rho \). Returning for a minute to our charged
sphere problem, very close to its surface it may be considered planar, so that the electric field should indeed be continuous, as it is.

Admittedly, the integral form (16) of the Gauss law is immediately useful only for highly symmetrical geometries, like as in the two problems discussed above. However, it may be recast into an alternative, differential form whose field of useful applications is much wider. This form may be obtained from Eq. (16) using the divergence theorem that, according to the vector algebra, is valid for any space-differentiable vector, in particular \( \mathbf{E} \), and for any volume \( V \) limited by closed surface \( S \):

\[
\oint_S \mathbf{E} \cdot d\mathbf{A} = \int_V \nabla \cdot \mathbf{E} \, dV,
\]

where \( \nabla \) is the del (or “nabla”) operator of spatial differentiation. Combining Eq. (25) with the Gauss law (16), we get

\[
\int_V \left( \nabla \cdot \mathbf{E} - \frac{\rho}{\varepsilon_0} \right) \, dV = 0.
\]

For a given distribution of electric charge (and hence of the electric field), this equation should be valid for any choice of volume \( V \). This can hold only if the function under the integral vanishes at each point, i.e. if

\[
\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.
\]

(1.27)

Note that in a sharp contrast with the integral form (16), Eq. (26) is local: it relates the electric field divergence to the charge density at the same point. This equation, being the differential form of the Gauss law, is frequently called (the free-space version of) one of Maxwell equations. Another, homogeneous Maxwell equation’s “embryo” may be obtained by noticing that curl of point charge’s field, and hence that of any system of charges, equals zero:

\[
\nabla \times \mathbf{E} = 0.
\]

(1.28)

(We will arrive at two other Maxwell equations, for the magnetic field, in Chapter 5, and then generalize all the equations to their full, time-dependent form by the end of Chapter 6. However, Eq. (27) would stay the same.)

Just to get a better gut feeling of Eq. (27), let us apply it to the same example of a uniformly charged sphere (Fig. 2). The vector algebra tells us that the divergence of a spherically symmetric vector function \( \mathbf{E}(r) = E(r) \mathbf{n} \), may be simply expressed in spherical coordinates:

\[
13 \text{ See, e.g., MA Eq. (12.2). Note that the scalar product under the integral in Eq. (25) is nothing else than the divergence of vector } \mathbf{E} \text{ – see, e.g., MA Eq. (8.4).}
\]

\[14 \text{ See, e.g., MA Secs. 8-10.}
\]

\[15 \text{ In the Gaussian units, just as in the initial Eq. (5), } \varepsilon_0 \text{ has to be replaced with } 1/4\pi, \text{ so that the Maxwell equation (27) looks like } \nabla \cdot \mathbf{E} = 4\pi \rho, \text{ while Eq. (28) stays the same.}
\]

\[16 \text{ This follows, for example, from the direct application of MA Eq. (10.11) to the spherically-symmetric vector function } \mathbf{f} = \mathbf{E}(r) = E(r) \mathbf{n}, \text{ field of a point charge placed at the origin, giving } f_\phi = f_\varphi = 0 \text{ and } \partial f_\phi/\partial \theta = \partial f_\varphi/\partial \phi = 0.
\]

\[17 \text{ See, e.g., MA Eq. (10.10) for this particular case (when } \partial/\partial \theta = \partial/\partial \varphi = 0).}
\]
\begin{equation}
\n\nabla \cdot \mathbf{E} = \frac{1}{r^2} \frac{d}{dr} (r^2 E). \tag{1.29}
\end{equation}

As a result, Eq. (27) yields a linear, ordinary differential equation for the function \(E(r)\):

\[ \frac{1}{r^2} \frac{d}{dr} (r^2 E) = \begin{cases} 
\rho / \varepsilon_0, & \text{for } r \leq R, \\
0, & \text{for } r \geq R,
\end{cases} \tag{1.30a} \]

that may be readily integrated on each of the segments:

\[ E(r) = \frac{1}{\varepsilon_0} \frac{1}{r^2} \int \left[ \rho \int r^2 dr = \rho r^3 / 3 + C_1, \right. \left. \text{for } r \leq R, \right. \left. C_2, \right. \left. \text{for } r \geq R. \right. \tag{1.30b} \]

In order to determine the integration constant \(C_1\), we can use boundary condition \(E(0) = 0\). (It follows from problem’s spherical symmetry: in the center of the sphere, electric field has to vanish, because otherwise, where would it be directed?) Constant \(C_2\) may be found from the continuity condition \(E(R - 0) = E(R + 0)\), which has already been discussed above. As a result, we arrive at our previous results (19) and (22).

We can see that in this particular, highly symmetric case, using the differential form of the Gauss law is more complex than its integral form. (For our second example, shown in Fig. 4, it would be even less natural.) However, Eq. (27) and its generalizations are more convenient for asymmetric charge distributions, and invaluable in the cases where the charge distribution \(\rho(r)\) is not known a priori and has to be found in a self-consistent way. (We will start discussing such cases in the next chapter.)

### 1.3. Scalar potential and electric field energy

One more help for solving electrostatics (and more complex) problems may be obtained from the notion of the electrostatic potential, which is just the electrostatic potential energy \(U\) of a probe particle, normalized by its charge:

\[ \phi \equiv \frac{U}{q}. \tag{1.31} \]

As we know from classical mechanics,\(^{18}\) the notion of \(U\) (and hence \(\phi\)) make sense only for the case of potential forces, for example those depending just on particle’s position. Equations (6) and (8) show that, in the static situations, the electric field clearly falls into this category. For such a field, the potential energy may be defined as a scalar function \(U(r)\) that allows the force to be calculated as its gradient (with the opposite sign):

\[ \mathbf{F} = -\nabla U. \tag{1.32} \]

Dividing both sides of this equation by the charge of the probe particle, and using Eqs. (5) and (31), we get\(^{19}\)

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\(^{18}\) See, e.g., CM Sec. 1.4.

\(^{19}\) Eq. (28) could be also derived from this relation, because according to vector algebra, any gradient field has vanishing curl - see, e.g., MA Eq. (11.1).
In order to calculate the scalar potential, let us start from the simplest case of a single point charge \( q \) placed at the origin. For it, the Coulomb law (5) takes a simple form

\[
E = \frac{q}{4\pi\varepsilon_0} \frac{r}{r^3} = \frac{n}{4\pi\varepsilon_0} \frac{r}{r^2} .
\]  

(1.34)

It is straightforward to check that the last fraction in the right-hand part of this equation is equal to \(-\nabla(1/r)\).\(^{20}\) Hence, according to the definition (33), for this particular case

\[
\phi = \frac{1}{4\pi\varepsilon_0} \frac{q}{r} .
\]  

(1.35)

(In the Gaussian units, this result is spectacularly simple: \( \phi = q/r \).) Note that we could add an arbitrary constant to this potential (and indeed to any other distribution of \( \phi \) discussed below) without changing the force, but it is convenient to define the potential energy to approach zero at infinity.

Before going any further, let us demonstrate how useful the notions of \( U \) and \( \phi \) are, on a very simple example. Let two similar charges \( q \) be launched from afar, with an initial velocity \( v_0 \ll c \) each, straight toward each other (i.e. with the zero impact parameter) – see Fig. 5. Since, according to the Coulomb law, the charges repel each other with increasing force, they will stop at some minimum distance \( r_{\text{min}} \) from each other, and than fly back.

![Fig. 1.5. Simple problem of electric particle motion.](image)

We could of course find \( r_{\text{min}} \) directly from the Coulomb law. However, for that we would need to write the 2\(^{\text{nd}}\) Newton law for each particle (actually, due to the problem symmetry, they would be similar), then integrate them over time once to find the particle velocity \( v \) as a function of distance, and then recover \( r_{\text{min}} \) from the requirement \( v = 0 \). The notion of potential allows this problem to be solved in one line. Indeed, in the field of potential forces the system’s total energy \( E = T + U = T + q\phi \) is conserved. In our non-relativistic case, the kinetic energy \( T \) is just \( mv^2/2 \). Hence, equating the total energy of two particles in the points \( r = \infty \) and \( r = r_{\text{min}} \), and using Eq. (35) for \( \phi \), we get

\[
2 \frac{mv_0^2}{2} + 0 = 0 + \frac{1}{4\pi\varepsilon_0} \frac{q^2}{r_{\text{min}}} ,
\]  

(1.36)

immediately giving us the final answer: \( r_{\text{min}} = q^2/(4\pi\varepsilon_0 m v_0^2) \).

Now let us calculate \( \phi \) for an arbitrary configuration of charges. For a single charge in an arbitrary position (say, \( r_k \)), \( r \) in Eq. (35) should be evidently replaced for \( |r - r_k| \). Now, the linear

\(20\) This may be done either by Cartesian components or using the well-known expression \( \nabla f = (df/dr) n \), valid for any spherically-symmetric scalar function \( f(r) \) - see, e.g., MA Eq. (10.8) for the particular case \( \partial/\partial \theta = \partial/\partial \phi = 0 \).
superposition principle (3) allows for an easy generalization of this formula to the case of an arbitrary set of discrete charges,

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{k \neq k'} \frac{q_k}{|\mathbf{r} - \mathbf{r}_k|}. \quad (1.37)$$

Finally, using the same arguments as in Sec. 1, we can use this result to argue that in the case of an arbitrary continuous charge distribution

$$\phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'. \quad (1.38)$$

Again, the notion of Dirac’s delta-function allows to use the last equation for discrete charges as well, so that Eq. (38) may be considered as the general expression for the electrostatic potential.

For most practical calculations, using this expression and then applying Eq. (33) to the result, is preferable to using Eq. (9), because $\phi$ is a scalar, while $\mathbf{E}$ is a 3D vector - mathematically equivalent to 3 scalars. Still, this approach may lead to technical problems similar to those discussed in Sec. 2. For example, applying it to the spherically-symmetric distribution of charge (Fig. 2), we get integral

$$\phi = \frac{1}{4\pi\varepsilon_0} \int_0^\pi \sin \theta' d\theta' \int_0^{2\pi} r'^2 dr' \rho(\mathbf{r}') \cos \theta', \quad (1.39)$$

which is not much simpler than Eq. (11).

The situation may be much improved by re-casting Eq. (38) into a differential form. For that, it is sufficient to plug the definition of $\phi$, Eq. (33), into Eq. (27):

$$\nabla \cdot (-\nabla \phi) = \frac{\rho}{\varepsilon_0}. \quad (1.40)$$

The left-hand part of this equation is nothing more than the Laplace operator of $\phi$ (with the minus sign), so that we get the famous Poisson equation\textsuperscript{21} for the electrostatic potential:

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_0}. \quad (1.41)$$

(In the Gaussian units, the Poisson equation looks like $\nabla^2 \phi = -4\pi\rho$.) This differential equation is so convenient for applications that even its particular case for $\rho = 0$,

$$\nabla^2 \phi = 0, \quad (1.42)$$

has earned a special name – the Laplace equation\textsuperscript{22}.

In order to get a feeling of the Poisson equation as a problem solving tool, let us return to the spherically-symmetric charge distribution (Fig. 2) with a constant charge density $\rho$. Using the

\textsuperscript{21} Named after S. D. Poisson (1781-1840), also famous for the Poisson distribution – one of the central results of the probability theory - see, e.g., SM Sec. 5.2.

\textsuperscript{22} After mathematician (and astronomer) P. S. de Laplace (1749-1827) who, together with A. Clairault, is credited for the development of the very concept of potential.
symmetry, we can present the potential as \( \phi(r) = \phi(r) \), and hence use the following simple expression for its Laplace operator:\(^{23}\)

\[
\nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right),
\]

so that for the points inside the charged sphere \((r \leq R)\) the Poisson equation yields

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = -\frac{\rho}{\varepsilon_0}, \quad \text{i.e.} \quad \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = -\frac{\rho}{\varepsilon_0} r^2.
\]

(1.44)

Integrating the last form of the equation over \(r\) once, with the natural boundary condition \(\frac{d\phi}{dr} \mid_{r=0} = 0\) (because of the condition \(E(0) = 0\), which has been discussed above), we get

\[
\frac{d\phi}{dr}(r) = -\frac{\rho}{r^2 \varepsilon_0} \int_0^r r'^2 dr' = -\frac{\rho r}{3 \varepsilon_0} = -\frac{1}{4\pi\varepsilon_0} \frac{Qr}{R^3}.
\]

(1.45)

Since this derivative is nothing more than \(-E(r)\), in this formula we can readily recognize our previous result (22). Now we may like to carry out the second integration to calculate the potential itself:

\[
\phi(r) = -\frac{Q}{4\pi\varepsilon_0 R^3} \int_0^r r'dr' + c_1 = -\frac{Qr^2}{8\pi\varepsilon_0 R^3} + c_1,
\]

(1.46)

Before making any judgment on the integration constant \(c_1\), let us solve the Poisson equation (in this case, just the Laplace equation) for the range outside the sphere \((r > R)\):

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = 0.
\]

(1.47)

Its first integral,

\[
\frac{d\phi}{dr}(r) = \frac{c_2}{r^2},
\]

also gives the electric field (with the minus sign). Now using Eq. (1.45) and requiring the field to be continuous at \(r = R\), we get

\[
\frac{c_2}{R^2} = -\frac{Q}{4\pi\varepsilon_0 R^2}, \quad \text{i.e.} \quad \frac{d\phi}{dr}(r) = -\frac{Q}{4\pi\varepsilon_0 r^2},
\]

(1.49)

in an evident agreement with Eq. (19). Integrating this result again,

\[
\phi(r) = -\frac{Q}{4\pi\varepsilon_0} \int \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_0 r} + c_3, \quad \text{for} \quad r > R,
\]

(1.50)

we can select \(c_3 = 0\), so that \(\phi(\infty) = 0\), in accordance with the usual (though not compulsory) convention. Now we can finally determine constant \(c_1\) in Eq. (46) by requiring that this equation and Eq. (50) give the same value of \(\phi\) at the boundary \(r = R\). (According to Eq. (33), if the potential had a jump, the electric field at that point would be infinite.) The final answer may be presented as

\[^{23}\text{See, e.g., MA Eq. (10.8) for } \partial/\partial \theta = \partial/\partial \phi = 0.\]
\[ \phi(r) = \frac{Q}{4\pi\varepsilon_0 R} \left[ \frac{R^2 - r^2}{2R^2} + 1 \right], \quad \text{for } r \leq R. \quad (1.51) \]

We see that using the Poisson equation to find the electrostatic potential distribution for highly symmetric problems may be more cumbersome than directly finding the electric field – say, from the Gauss law. However, we will repeatedly see below that if the electric charge distribution is not fixed in advance, using Eq. (41) may be the only practicable way to proceed.

Returning now to the general theory of electrostatic phenomena, let us calculate potential energy \( U \) of an arbitrary system of electric charges \( q_k \). Despite the apparently straightforward relation (31) between \( U \) and \( \phi \), the calculation is a little bit more complex than one might think. Indeed, let us rewrite Eqs. (32), (33) for a single charge in the integral form:

\[ U(r) = -\int_{r_0}^{r} F(r') \cdot dr', \quad \text{i.e.} \quad \phi(r) = -\int_{r_0}^{r} E(r') \cdot dr', \quad (1.52) \]

where \( r_0 \) is some reference point. These integrals reflect the fact that the potential energy is just the work necessary to move the charge from point \( r_0 \) to point \( r \), and clearly depend on whether the charge motion affects force \( F \) (and hence electric field \( E \)) or not. If it does not, i.e. if the field is produced by some external charges (such fields \( E_{\text{ext}} \) are also called external), everything is simple indeed: using the linearity of relations (31) and (32), for the total potential energy we may write

\[ U_{\text{ext}} = \sum_k q_k \phi_{\text{ext}}(r_k), \quad \text{where} \quad \phi_{\text{ext}}(r) \equiv -\int_{r_0}^{r} E_{\text{ext}}(r') \cdot dr'. \quad (1.53) \]

Repeating the argumentation that has led us to Eq. (9), we see that for a continuously distributed charge, this sum turns into an integral:

\[ U_{\text{ext}} = \int \rho(r) \phi_{\text{ext}}(r) d^3r. \quad (1.54) \]

However, if the electric field is created by the charges whose energy we are calculating, the situation is somewhat different. To calculate \( U \) for this case, let us use the fact its independence of the way the charge configuration has been created, considering the following process. First, let us move one charged particle (say, \( q_1 \)) from infinity to an arbitrary point of space \( (r_1) \) in the absence of other charges. During the motion the particle does not experience any force (again, the charge does not interact with itself!), so that its potential energy is the same as at infinity (with the standard choice of the arbitrary constant, zero): \( U_1 = 0 \). Now let us fix the position of that charge, and move another charge \( (q_2) \) from infinity to point \( r_2 \) (with velocity \( v << c \), in order to avoid any magnetic field effects, to be discussed in Chapter 5.) This particle, during its motion, does experience the Coulomb force exerted by fixed \( q_1 \), so that according to Eq. (31), its contribution to the final potential energy

\[ U_2 = q_2 \phi(r_2). \quad (1.55) \]

Since the first particle was not moving during this process, the total potential energy \( U \) of the system is equal to just \( U_2 \). This is exactly the equality used for writing the right-hand part of Eq. (36). (Prescribing a similar energy to charge \( q_1 \) as well would constitute an error – a very popular one, and hence having a special name, double-counting.)
Now, fixing the first two charges in points \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \), respectively, and bringing in the third charge from infinity, we increment the potential energy by

\[
U_3 = q_3 \left[ \phi_1 (\mathbf{r}_3) + \phi_2 (\mathbf{r}_3) \right].
\]  

(1.56)

I believe that at this stage it is already clear how to generalize this result to the contribution from an arbitrary \((k-\text{th})\) charge being moved in (Fig. 6):

\[
U_k = q_k \left[ \phi_1 (\mathbf{r}_k) + \phi_2 (\mathbf{r}_k) + \cdots + \phi_{k-1} (\mathbf{r}_k) \right] = q_k \sum_{k' < k} \phi_{k'} (\mathbf{r}_k).
\]  

(1.57)

(Notice condition \( k' \leq k \), which suppresses erroneous double-counting.)

Now, summing up all the increments, for the total electrostatic energy of the system we get:

\[
U = \sum_k U_k = \sum_{k, k'} q_k \phi_{k'} (\mathbf{r}_k).
\]  

(1.58)

This is our final result in its generic form; it is so important that it is worthy of rewriting it in two other forms. First, for its generalization to the continuous charge distribution, we may use Eq. (35) to present Eq. (58) in a more symmetric form:

\[
U = \frac{1}{4 \pi \varepsilon_0} \sum_{k, k'} \frac{q_k q_{k'}}{|\mathbf{r}_k - \mathbf{r}_{k'}|}.
\]  

(1.59)

The expression under the sum is evidently symmetric with respect to the index swap, so that it may be rewritten in a fully symmetric form,

\[
U = \frac{1}{4 \pi \varepsilon_0} \frac{1}{2} \sum_{k, k'} \frac{q_k q_{k'}}{|\mathbf{r}_k - \mathbf{r}_{k'}|},
\]  

(1.60)

which is now easily generalized to the continuous case:

\[
U = \frac{1}{4 \pi \varepsilon_0} \frac{1}{2} \int d^3 r \int d^3 r' \frac{\rho (\mathbf{r}) \rho (\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}.
\]  

(1.61)

(As before, in this case the restriction expressed in the discrete charge case as \( k \neq k' \) is not important, because if the charge density is a continuous function, integral (61) does not diverge at point \( \mathbf{r} = \mathbf{r}' \).)
To present this result in one more form, let us notice that according to Eq. (38), the integral over \( r' \) in Eq. (61), divided by \( 4\pi\varepsilon_0 \), is just the full electrostatic potential at point \( r \), and hence

\[
U = \frac{1}{2} \int \rho(r)\phi(r)d^3r.
\]  

(1.62)

For the discrete charge case, this result becomes

\[
U = \frac{1}{2} \sum_k q_k\phi(r_k),
\]

(1.63)

but now it is important to remember that the "full" potential’s value \( \phi(r_k) \) should exclude the (infinite) contribution of charge \( k \) itself. Comparing the last two formulas with Eqs. (52) and (53), we see that the electrostatic energy of charge interaction, as expressed via the charge-potential product, is twice less than that of charge energy in a fixed ("external") field. This is evidently the result of the self-consistent build-up of the electric field as the charge system is being formed.24

Now comes an important conceptual question: can we locate this interaction energy in space? Expressions (60)-(63) seem to imply that contributions to \( U \) come only from the regions where electric charges are located. However, one of the beautiful features of physics is that sometimes completely different interpretations of the same mathematical result are possible. In order to get an alternative view at our current result, let us write Eq. (62) for a volume \( V \) so large that the electric field on the limiting surface \( A \) is negligible, and plug into it the charge density expressed from the Poisson equation (41):

\[
U = -\frac{\varepsilon_0}{2} \int_V \phi \nabla^2 \phi d^3r.
\]

(1.64)

This expression may be integrated by parts as\(^{25}\)

\[
U = -\frac{\varepsilon_0}{2} \left[ \int_A \phi (\nabla \phi) \cdot d^2r - \int_V (\nabla \phi)^2 d^3r \right].
\]

(1.65)

According to our condition of negligible field \( E = -\nabla \phi \) on the surface, the first integral vanishes, and we get a very important formula

\[
U = \frac{\varepsilon_0}{2} \int_V (\nabla \phi)^2 d^3r = \frac{\varepsilon_0}{2} \int E^2 d^3r.
\]

(1.66)

This result certainly invites an interpretation very much different than Eq. (62): it is natural to represent it in the following form:

\[
U = \int u(r)d^3r, \text{ with } u(r) = \frac{\varepsilon_0}{2} E^2(r),
\]

(1.67)

---

24 The nature of this additional factor \( \frac{1}{2} \) is absolutely the same as in the well-known formula \( U = (\frac{1}{2})kx^2 \) for the potential energy of an elastic spring providing returning force \( F = -kx \) proportional to the deviation \( x \) from equilibrium.

25 This transformation follows from the divergence theorem MA (12.2) applied to vector function \( \mathbf{f} = \phi \nabla \phi \), taking into account the 3D differentiation rule MA Eq. (11.4a): \( \nabla \cdot (\phi \nabla \phi) = (\nabla \phi) \cdot (\nabla \phi) + \phi \nabla \cdot (\nabla \phi) = (\nabla \phi)^2 + \phi \nabla^2 \phi. \)
and interpret $u(r)$ as the spatial density of the electric field energy, which is continuously distributed over all the space where the field exists - rather than just its part where the charges are located.

Let us have a look how these two alternative pictures work for our testbed problem, a uniformly charged sphere. If we start from Eq. (62), we may limit integration by the sphere volume ($0 \leq r \leq R$) where $\rho \neq 0$. Using Eq. (51), and the spherical symmetry of the problem ($d^3r = 4\pi r^2 dr$), we get

\[ U = \frac{1}{2} 4\pi \int_0^R \rho \phi r^2 dr = \frac{1}{2} 4\pi \rho \frac{Q}{4\pi \varepsilon_0 R} \int_0^R \left[ \frac{R^2 - r^2}{2R^2} + 1 \right] r^2 dr = \frac{1}{5} \frac{1}{4\pi \varepsilon_0} \frac{Q^2}{2}. \] (1.68)

On the other hand, if we use Eq. (67), we need to integrate energy everywhere, i.e. both inside and outside of the sphere:

\[ U = \frac{\varepsilon_0}{2} 4\pi \int_0^R E^2 r^2 dr + \int_0^\infty E^2 r^2 dr. \] (1.69)

Using Eqs. (19) and (22) for, respectively, the external and internal regions, we get

\[ U = \frac{\varepsilon_0}{2} 4\pi \int_0^R \left( \frac{Qr}{4\pi \varepsilon_0} \right)^2 r^2 dr + \int_0^\infty \left( \frac{Q}{4\pi \varepsilon_0 r^2} \right)^2 r^2 dr = \left( \frac{1}{5} + 1 \right) \frac{1}{4\pi \varepsilon_0} \frac{Q^2}{2}. \] (1.70)

This is (fortunately :-) the same answer as given by Eq. (68), but to some extent it is more informative because it shows how exactly the electric field energy is distributed between the interior and exterior of the charged sphere.27

We see that, as we could expect, within the realm of electrostatics, Eqs. (62) and (67) are equivalent. However, when we examine electrodynamics in Chapter 6 and on, we will see that the latter equation is more general, and that it is more adequate to associate energy with the field itself rather than its sources - in our current case, electric charges.

### 1.4. Exercise problems

1.1. Calculate the electric field created by a thin, long, straight filament, electrically charged with a constant linear density $\lambda$, using two approaches:
   (i) directly from the Coulomb law, and
   (ii) using the Gauss law.

1.2. Two thin, straight parallel filaments, separated by distance $\rho$, carry equal and opposite uniformly distributed charges with linear density $\lambda$ - see Fig. on the right. Calculate the electrostatic force (per unit length) of the Coulomb

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26 In the Gaussian units, the standard replacement $\varepsilon_0 \rightarrow 1/4\pi$ turns the last of Eqs. (67) into $u(r) = E^2/8\pi$.

27 Note that $U \rightarrow \infty$ at $R \rightarrow 0$. Such divergence appears at application of Eq. (67) to any point charge. Since it does not affect the force acting on the charge, the divergence does not create any technical difficulty for analysis of charge statics or non-relativistic dynamics, but it points to a conceptual problem of classical electrodynamics as the whole. This issue will be discussed in the very end of the course (Sec. 10.6).
interaction between the wires. Compare the result with the Coulomb law for the force between the point charges, and interpret their difference.

1.3. A sphere of radius $R$, whose volume had been charged with a constant density $\rho$, is split with a very narrow, planar gap passing through its center. Find the Coulomb force between the resulting two hemispheres.

1.4. Calculate the distribution of the electrostatic potential created by a straight, thin filament of finite length $2l$, charged with a constant linear density $\lambda$, and explore the result in the limits of very small and very large distances from the filament.

1.5. A thin plane sheet, perhaps of an irregular shape, carries an electric charge distributed over the sheet with a constant areal density $\sigma$.

(i) Express the electric field component normal to the plane, at a certain distance from it, via the solid angle $\Omega$ at which the sheet is visible from the observation point.

(ii) Use the result to calculate the field in the center of a cube, with one face charged with constant density $\sigma$.

1.6. Can one create electrostatic fields with the Cartesian components proportional to the following products of Cartesian coordinates $\{x, y, z\}$,

(i) $\{yz, xz, xy\}$,

(ii) $\{xy, yz, xz\}$,

in a finite region of space?

1.7. Distant sources have been used to create different electric fields on two sides of a wide and thin metallic membrane with a round hole of radius $R$ in it - see Fig. on the right. Besides the local perturbation created by the hole, the fields are uniform:

$$E_{\rho >> R} = n_z \times \begin{cases} E_1, & \text{at } z < 0, \\ E_2, & \text{at } z > 0. \end{cases}$$

Prove that the system may serve as an electrostatic lens for charged particles flying along axis $z$, at distances $\rho << R$ from it, and calculate the focal distance $f$ of the lens. Spell out the conditions of validity of your result.

1.8. By direct calculation, find the average electric potential of the spherical surface of radius $R$, created by a point charge $q$ located at distance $r > R$ from the sphere’s center. Use the result to prove the following general mean value theorem: the electric potential at any point is always equal to its average value on any spherical surface with the center at that point, and containing no electric charges inside it.

1.9. Calculate the electrostatic energy per unit area of the system of two thin, parallel planes with equal and opposite charges of a constant areal density $\sigma$, separated by distance $d$ - see Fig. on the right.
1.10. The system analyzed in the previous problem (two thin, parallel, oppositely charged planes) is now placed into an external, uniform, normal electric field \( E_{\text{ext}} = \frac{\sigma}{\varepsilon_0} \) – see Fig. on the right. Find the forces (per unit area) acting on each plane, by two methods:
(i) directly from the electric field distribution, and
(ii) from the potential energy of the system.

1.11. A thin spherical shell of radius \( R \), which had been charged with a constant areal density \( \sigma \), is split into two equal halves by a very narrow, planar cut passing through sphere’s center. Calculate the force of electrostatic repulsion between the resulting hemispheric shells.

1.12. Two similar thin, circular, coaxial disks of radius \( R \), separated by distance \( 2d \), are uniformly charged with equal and opposite areal densities \( \pm \sigma \) - see Fig. on the right. Calculate and sketch the distribution of the electrostatic potential and the electric field of the disks along their common axis.

1.13. In a certain reference frame, the electrostatic potential created by some electric charge distribution, is
\[
\phi(r) = C \left( \frac{1}{r} + \frac{1}{2r_0} \right) \exp \left\{ \frac{-r}{r_0} \right\},
\]
where \( C \) and \( r_0 \) are constants, and \( r \equiv |r| \) is the distance from the origin. Calculate the charge distribution in space.

1.14. A thin flat sheet, cut in a form of a rectangle of size \( a \times b \), is electrically charged with a constant areal density \( \sigma \). Without an explicit calculation of the spatial distribution \( \phi(r) \) of the electrostatic potential induced by this charge, find the ratio of its values at the center and at the corners of the rectangle.

\textit{Hint:} Consider partitioning the rectangle into several similar parts and using the linear superposition principle.

1.15. Explore the relation between the Laplace equation (42) and the condition of minimum of the electrostatic field energy (67).

1.16. Calculate the energy of electrostatic interaction of two spheres, of radii \( R_1 \) and \( R_2 \), each with a spherically-symmetric charge distribution, separated by distance \( d > R_1 + R_2 \).
1.17. Prove the following *reciprocity theorem of electrostatics*: if two spatially-confined charge distributions \( \rho_1(\mathbf{r}) \) and \( \rho_2(\mathbf{r}) \) create respective distributions \( \phi_1(\mathbf{r}) \) and \( \phi_2(\mathbf{r}) \) of the electrostatic potential, then

\[
\int \rho_1(\mathbf{r}) \phi_2(\mathbf{r}) \, d^3r = \int \rho_2(\mathbf{r}) \phi_1(\mathbf{r}) \, d^3r.
\]

*Hint:* Consider integral \( \int \mathbf{E}_1 \cdot \mathbf{E}_2 \, d^3r \).

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28 This is only the simplest one of the whole family of reciprocity theorems in electromagnetism. (Sometimes it is called "Green's reciprocity theorem", but historically it is more fair to reserve the last name for the generalization to surface charges, using Eq. (2.210), to be discussed in Sec. 2.7 below.)