Chapter 4. Rigid Body Motion

This chapter discusses the motion of rigid bodies, with a heavy focus on its most nontrivial part: the rotation. Some byproduct results of this analysis enable a discussion, at the end of the chapter, of the motion of point particles as observed from non-inertial reference frames.

4.1. Translation and rotation

It is natural to start a discussion of many-particle systems from a (relatively :-) simple limit when the changes of distances $r_{kk'} \equiv |r_k - r_{k'}|$ between the particles are negligibly small. Such an abstraction is called the (absolutely) rigid body, and is a reasonable approximation in many practical problems, including the motion of solids. In other words, this model neglects deformations – which will be the subject of the next chapters. The rigid-body approximation reduces the number of degrees of freedom of the system of $N$ particles from $3N$ to just six – for example, three Cartesian coordinates of one point (say, 0), and three angles of the system’s rotation about three mutually perpendicular axes passing through this point – see Fig. 1.\(^1\)

As it follows from the discussion in Secs. 1.1-3, any purely translational motion of a rigid body, at which the velocity vectors $v$ of all points are equal, is not more complex than that of a point particle. Indeed, according to Eqs. (1.8) and (1.30), in an inertial reference frame such a body moves, upon the effect of the net external force $\mathbf{F}^{(\text{ext})}$, exactly as a point particle. However, the rotation is a bit more tricky.

Let us start by showing that an arbitrary elementary displacement of a rigid body may be always considered as a sum of the translational motion discussed above, and what is called a pure rotation. For that, consider a “moving” reference frame, firmly bound to the body, and an arbitrary vector $\mathbf{A}$ (Fig. 1). The vector may be represented by its Cartesian components $A_j$ in that moving frame:

$$\mathbf{A} = \sum_{j=1}^{3} A_j \mathbf{n}_j .$$

\(^1\) An alternative way to arrive at the same number six is to consider three points of the body, which uniquely define its position. If movable independently, the points would have nine degrees of freedom, but since three distances $r_{kk'}$ between them are now fixed, the resulting three constraints reduce the number of degrees of freedom to six.
Let us calculate the time derivative of this vector as observed from a different (“lab”) frame, taking into account that if the body rotates relative to this frame, the directions of the unit vectors $\mathbf{n}_j$ as seen from the lab frame, change in time. Hence, we have to differentiate both operands in each product contributing to the sum (1):

$$\frac{dA}{dt}_{\text{in\,lab}} = \sum_{j=1}^{3} \frac{dA_j}{dt} \mathbf{n}_j + \sum_{j=1}^{3} A_j \frac{d\mathbf{n}_j}{dt}. \quad (4.2)$$

On the right-hand side of this equality, the first sum evidently describes the change of vector $\mathbf{A}$ as observed from the moving frame. In the second sum, each of the infinitesimal vectors $d\mathbf{n}_j$ may be represented by its Cartesian components:

$$d\mathbf{n}_j = \sum_{j'=1}^{3} d\varphi_{jj'} \mathbf{n}_{j'}. \quad (4.3)$$

where $d\varphi_{jj'}$ are some dimensionless scalar coefficients. To find out more about them, let us scalar-multiply each side of Eq. (3) by an arbitrary unit vector $\mathbf{n}_{j''}$, and take into account the evident orthonormality condition:

$$\mathbf{n}_j \cdot \mathbf{n}_{j'} = \delta_{j''}, \quad (4.4)$$

where $\delta_{j''}$ is the Kronecker delta symbol. As a result, we get

$$d\mathbf{n}_j \cdot \mathbf{n}_{j'} = d\varphi_{jj'}. \quad (4.5)$$

Now let us use Eq. (5) to calculate the first differential of Eq. (4):

$$d\mathbf{n}_j \cdot \mathbf{n}_{j'} + \mathbf{n}_j \cdot d\mathbf{n}_{j'} = d\varphi_{jj'} + d\varphi_{jj'} = 0; \quad \text{in particular,} \quad 2d\mathbf{n}_j \cdot \mathbf{n}_j = 2d\varphi_{jj} = 0. \quad (4.6)$$

These relations, valid for any choice of indices $j, j'$, and $j''$ of the set $\{1, 2, 3\}$, show that the matrix of $d\varphi_{jj'}$ is antisymmetric with respect to the swap of its indices; this means that there are not nine just three non-zero independent coefficients $d\varphi_{jj'}$, all with $j \neq j'$. Hence it is natural to renumber them in a simpler way: $d\varphi_{jj'} = -d\varphi_{j'j} = d\varphi_{jj''}$, where the indices $j, j'$, and $j''$ follow in the “correct” order – either $\{1,2,3\}$, or $\{2,3,1\}$, or $\{3,1,2\}$. It is easy to check (either just by a component-by-component comparison or using the Levi-Civita permutation symbol $\varepsilon_{jj''}$) that in this new notation, Eq. (3) may be represented just as a vector product:

$$d\mathbf{n}_j = d\varphi \times \mathbf{n}_j, \quad (4.7)$$

where $d\varphi$ is the infinitesimal vector defined by its Cartesian components $d\varphi_j$ in the rotating reference frame $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ – see Eq. (3).

This relation is the basis of all rotation kinematics. Using it, Eq. (2) may be rewritten as

$$\frac{dA}{dt}_{\text{in\,lab}} = \frac{dA}{dt}_{\text{in\,mov}} + \sum_{j=1}^{3} A_j \frac{d\varphi}{dt} \times \mathbf{n}_j \equiv \frac{dA}{dt}_{\text{in\,mov}} + \mathbf{\omega} \times \mathbf{A}, \quad \text{where} \quad \mathbf{\omega} \equiv \frac{d\varphi}{dt}. \quad (4.8)$$

To reveal the physical sense of the vector $\mathbf{\omega}$, let us apply Eq. (8) to the particular case when $\mathbf{A}$ is the radius vector $\mathbf{r}$ of a point of the body, and the lab frame is selected in a special way: its origin has the

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2 See, e.g., MA Eq. (13.1).

3 See, e.g., MA Eq. (13.2). Using this symbol, we may write $d\varphi_{jj'} = -d\varphi_{j'j} \equiv \varepsilon_{jj''}d\varphi_j$ for any choice of $j, j'$, and $j''$. 
same position and moves with the same velocity as that of the moving frame in the particular instant under consideration. In this case, the first term on the right-hand side of Eq. (8) is zero, and we get

\[
\frac{dr}{dt} \bigg|_{\text{special frame}} = \omega \times r, \quad (4.9)
\]

were vector \( r \) itself is the same in both frames. According to the vector product definition, the particle velocity described by this formula has a direction perpendicular to the vectors \( \omega \) and \( r \) (Fig. 2), and magnitude \( \omega r \sin \theta \). As Fig. 2 shows, the last expression may be rewritten as \( \omega \rho \), where \( \rho = r \sin \theta \) is the distance from the line that is parallel to the vector \( \omega \) and passes through the point 0. This is of course just the pure rotation about that line (called the instantaneous axis of rotation), with the angular velocity \( \omega \). Since according to Eqs. (3) and (8), the angular velocity vector \( \omega \) is defined by the time evolution of the moving frame alone, it is the same for all points \( r \), i.e. for the rigid body as a whole. Note that nothing in our calculations forbids not only the magnitude but also the direction of the vector \( \omega \), and thus of the instantaneous axis of rotation, to change in time (and in many cases it does); hence the name.

![Figure 4.2. The instantaneous axis and the angular velocity of rotation.](image)

Now let us generalize our result a step further, considering two reference frames that do not rotate versus each other: one (“lab”) frame arbitrary, and another one selected in the special way described above, so that for it Eq. (9) is valid in it. Since their relative motion of these two reference frames is purely translational, we can use the simple velocity addition rule given by Eq. (1.6) to write

\[
v \bigg|_{\text{lab}} = v_0 \bigg|_{\text{lab}} + v \bigg|_{\text{special frame}} = v_0 \bigg|_{\text{lab}} + \omega \times r, \quad (4.10)
\]

where \( r \) is the radius vector of a point is measured in the body-bound (“moving”) frame 0.

### 4.2. Inertia tensor

Since the dynamics of each point of a rigid body is strongly constrained by the conditions \( r_{kk'} = \text{const} \), this is one of the most important fields of application of the Lagrangian formalism discussed in Chapter 2. For using this approach, the first thing we need to calculate is the kinetic energy of the body in an inertial reference frame. Since it is just the sum of the kinetic energies (1.19) of all its points, we can use Eq. (10) to write:\(^{4}\)

\[
T \equiv \sum m \frac{v^2}{2} = \frac{m}{2} (v_0 + \omega \times r)^2 = \frac{m}{2} v_0^2 + \sum m v_0 \cdot (\omega \times r) + \frac{m}{2} (\omega \times r)^2. \quad (4.11)
\]

\(^{4}\)Actually, all symbols for particle masses, coordinates, and velocities should carry the particle’s index, over which the summation is carried out. However, in this section, for the notation simplicity, this index is just implied.
Let us apply to the right-hand side of Eq. (11) two general vector analysis formulas listed in the Math Appendix: the so-called operand rotation rule MA Eq. (7.6) to the second term, and MA Eq. (7.7b) to the third term. The result is

\[ T = \frac{m}{2} v_0^2 + \sum m \mathbf{r} \cdot (v_0 \times \mathbf{\omega}) + \sum \frac{m}{2} (\mathbf{\omega} \cdot \mathbf{r})^2. \]  

(4.12)

This expression may be further simplified by making a specific choice of the point 0 (from that the radius vectors \( \mathbf{r} \) of all particles are measured), namely by using for this point the center of mass of the body. As was already mentioned in Sec. 3.4 for the 2-point case, the radius vector \( \mathbf{R} \) of this point is defined as

\[ \mathbf{M} \mathbf{R} = \sum m \mathbf{r}, \quad \mathbf{M} = \sum m, \]  

(4.13)

where \( \mathbf{M} \) is the total mass of the body. In the reference frame centered as this point, \( \mathbf{R} = 0 \), so that the second sum in Eq. (12) vanishes, and the kinetic energy is a sum of just two terms:

\[ T = T_{\text{trans}} + T_{\text{rot}}, \quad T_{\text{trans}} = \frac{\mathbf{M}}{2} \mathbf{V}^2, \quad T_{\text{rot}} = \sum \frac{m}{2} (\mathbf{\omega} \cdot \mathbf{r})^2, \]  

(4.14)

where \( \mathbf{V} = d\mathbf{R}/dt \) is the center-of-mass velocity in our inertial reference frame, and all particle positions \( \mathbf{r} \) are measured in the center-of-mass frame. Since the angular velocity vector \( \mathbf{\omega} \) is common for all points of a rigid body, it is more convenient to rewrite the rotational energy in a form in that the summation over the components of this vector is clearly separated from the summation over the points of the body:

\[ T_{\text{rot}} = \frac{1}{2} \sum_{j=1}^{3} I_{jj} \omega_j \mathbf{\omega}_j, \]  

(4.15)

where the 3\times3 matrix with elements

\[ I_{jj} = \sum m (r_j^2 \delta_{jj} - r_j r_j) \]  

(4.16)

is called the inertia tensor of the body.\(^5\)

Actually, the term “tensor” for this matrix has to be justified, because in physics this term implies a certain reference-frame-independent notion, whose elements have to obey certain rules at the transfer between reference frames. To show that the matrix (16) indeed deserves its title, let us calculate another key quantity, the total angular momentum \( \mathbf{L} \) of the same body.\(^6\) Summing up the angular momenta of each particle, defined by Eq. (1.31), and then using Eq. (10) again, in our inertial reference frame we get

\[ \mathbf{L} = \sum \mathbf{r} \times \mathbf{p} = \sum m \mathbf{r} \times \mathbf{v} = \sum m \mathbf{r} \times (v_0 + \mathbf{\omega} \times \mathbf{r}) = \sum m \mathbf{r} \times v_0 + \sum m \mathbf{r} \times (\mathbf{\omega} \times \mathbf{r}). \]  

(4.17)

We see that the momentum may be represented as a sum of two terms. The first one,

\[^5\] While the ABCs of the rotational dynamics were developed by Leonhard Euler in 1765, an introduction of the inertia tensor’s formalism had to wait very long – until the invention of the tensor analysis by Tullio Levi-Civita and Gregorio Ricci-Curbastro in 1900 – soon popularized by its use in Einstein’s theory of general relativity.

\[^6\] Hopefully, there is very little chance of confusing the angular momentum \( \mathbf{L} \) (a vector) and its Cartesian components \( L_i \) (scalars with an index) on one hand, and the Lagrangian function \( L \) (a scalar without an index) on the other hand.
describes the possible rotation of the center of mass around the inertial frame’s origin. This term evidently vanishes if the moving reference frame’s origin 0 is positioned at the center of mass (where \( \mathbf{R} = 0 \)). In this case, we are left with only the second term, which describes a pure rotation of the body about its center of mass:

\[
\mathbf{L} = \mathbf{L}_{\text{rot}} \equiv \sum m \mathbf{r} \times (\mathbf{\omega} \times \mathbf{r}).
\]

(4.19)

Using one more vector algebra formula, the “bac minis cab” rule, we may rewrite this expression as

\[
\mathbf{L} = \sum m [(\mathbf{\omega} \cdot \mathbf{r})^2 \mathbf{r} - \mathbf{r} (\mathbf{\omega} \cdot \mathbf{r})].
\]

(4.20)

Let us spell out an arbitrary Cartesian component of this vector:

\[
L_j = \sum m \left[ \omega_j r_j^2 - r_j \sum_{j'=1}^{3} \omega_{j'} \right] = \sum m \sum_{j'=1}^{3} \omega_{j'} \left( r_j^2 \delta_{jj'} - r_j r_{j'} \right).
\]

(4.21)

By changing the summation order and comparing the result with Eq. (16), the angular momentum may be conveniently expressed via the same matrix elements \( I_{jj'} \) as the rotational kinetic energy:

\[
L_j = \sum_{j'=1}^{3} I_{jj'} \omega_{j'}.
\]

(4.22)

Since \( \mathbf{L} \) and \( \mathbf{\omega} \) are both legitimate vectors (meaning that they describe physical vectors independent of the reference frame choice), their connection, the matrix of elements \( I_{jj'} \), is a legitimate tensor. This fact, and the symmetry of the tensor \( (I_{jj'} = I_{jj}) \), which is evident from its definition (16), allow the tensor to be further simplified. In particular, mathematics tells us that by a certain choice of the axes’ orientations, any symmetric tensor may be reduced to a diagonal form

\[
I_{jj'} = I_j \delta_{jj'},
\]

where in our case

\[
I_j = \sum m \left( r_j^2 - r_j^2 \right) = \sum m \left( r_j^2 + r_j^2 \right) = \sum m \rho_j^2,
\]

(4.24)

\( \rho_j \) being the distance of the particle from the \( j^{th} \) axis, i.e. the length of the perpendicular dropped from the point to that axis. The axes of such a special coordinate system are called the principal axes, while the diagonal elements \( I_j \) given by Eq. (24), the principal moments of inertia of the body. In such a special reference frame, Eqs. (15) and (22) are reduced to very simple forms:

\[
T_{\text{rot}} = \sum_{j=1}^{3} \frac{I_j}{2} \omega_j^2,
\]

(4.25)

\[
L_j = I_j \omega_j.
\]

(4.26)

Both these results remind the corresponding relations for the translational motion, \( T_{\text{tran}} = MV^2/2 \) and \( P = MV \), with the angular velocity \( \mathbf{\omega} \) replacing the linear velocity \( \mathbf{V} \), and the tensor of inertia playing the role of scalar mass \( M \). However, let me emphasize that even in the specially selected reference frame, with

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7 See, e.g., MA Eq. (7.5).
axes pointing in principal directions, the analogy is incomplete, and rotation is generally more complex than translation, because the measures of inertia, \( I_j \), are generally different for each principal axis.

Let me illustrate this fact on a simple but instructive system of three similar massive particles fixed in the vertices of an equilateral triangle (Fig. 3).

Due to the symmetry of the configuration, one of the principal axes has to pass through the center of mass 0 and be normal to the plane of the triangle. For the corresponding principal moment of inertia, Eq. (24) readily yields \( I_3 = 3m\rho^2 \). If we want to express the result in terms of the triangle side \( a \), we may notice that due to the system’s symmetry, the angle marked in Fig. 3 equals \( \pi/6 \), and from the shaded right triangle, \( a/2 = \rho \cos(\pi/6) = \rho \sqrt{3}/2 \), giving \( \rho = a/\sqrt{3} \), so that, finally, \( I_3 = ma^2 \).

Let me use this simple case to illustrate the following general axis shift theorem, which may be rather useful – especially for more complex systems. For that, let us relate the inertia tensor components \( I_{jj'} \) and \( I'_{jj'} \), calculated in two reference frames – one with the origin in the center of mass 0, and another one (0’) displaced by a certain vector \( \mathbf{d} \) (Fig. 4a), so that for an arbitrary point, \( \mathbf{r}' = \mathbf{r} + \mathbf{d} \). Plugging this relation into Eq. (16), we get

\[
I'_{jj'} = \sum m \left[ (\mathbf{r} + \mathbf{d})^2 \delta_{jj'} - (r_j + d_j)(r_{j'} + d_{j'}) \right] = \sum m \left[ r^2 + 2\mathbf{r} \cdot \mathbf{d} + d^2 \right] \delta_{jj'} - (r_{j'} + r_j d_{j'} + r_j d_j + d_j d_{j'}). \tag{4.27}
\]

Since in the center-of-mass frame, all sums \( \sum m r_j \) equal zero, we may use Eq. (16) to finally obtain

\[
I'_{jj'} = I_{jj'} + M(\delta_{jj'} d^2 - d_{j'}d_j). \tag{4.28}
\]

In particular, this equation shows that if the shift vector \( \mathbf{d} \) is perpendicular to one (say, \( j^\text{th} \)) of the principal axes (Fig. 4b), i.e. \( d_j = 0 \), then Eq. (28) is reduced to a very simple formula:

\[
I'_{j} = I_{j} + Md^2. \tag{4.29}
\]
Now returning to the system shown in Fig. 3, let us perform such a shift to the new (“primed”) axis passing through the location of one of the particles, still perpendicular to the particles’ plane. Then the contribution of that particular mass to the primed moment of inertia vanishes, and \( I'_3 = 2ma^2 \). Now, returning to the center of mass and applying Eq. (29), we get
\[
I_3 = I'_3 - M\rho^2 = 2ma^2 - (3m)(a/\sqrt{3})^2 = ma^2, 
\]
i.e. the same result as above.

The symmetry situation inside the triangle’s plane is somewhat less evident, so let us start with calculating the moments of inertia for the axes shown vertical and horizontal in Fig. 3. From Eq. (24) we readily get:
\[
I_1 = 2mh^2 + m\rho^2 = m\left[2\left(\frac{a}{2\sqrt{3}}\right)^2 + \left(\frac{a}{\sqrt{3}}\right)^2\right] = \frac{ma^2}{2}, \quad I_2 = 2m\left(\frac{a}{2}\right)^2 = \frac{ma^2}{2}, \quad (4.30)
\]
where \( h \) is the distance from the center of mass and any side of the triangle: \( h = \rho \sin (\pi/6) = \rho/2 = a/2\sqrt{3} \). We see that \( I_1 = I_2 \), and mathematics tells us that in this case any in-plane axis (passing through the center-of-mass 0) may be considered as principal, and has the same moment of inertia. A rigid body with this property, \( I_1 = I_2 \neq I_3 \), is called the symmetric top. (The last direction is called the main principal axis of the system.)

Despite the symmetric top’s name, the situation may be even more symmetric in the so-called spherical tops, i.e. highly symmetric systems whose principal moments of inertia are all equal,
\[
I_1 = I_2 = I_3 = I, \quad (4.31)
\]
Mathematics says that in this case, the moment of inertia for rotation about any axis (but still passing through the center of mass) is equal to the same \( I \). Hence Eqs. (25) and (26) are further simplified for any direction of the vector \( \omega \):
\[
T_{\text{rot}} = \frac{I}{2}\omega^2, \quad \mathbf{L} = I\omega, \quad (4.32)
\]
thus making the analogy of rotation and translation complete. (As will be discussed in the next section, this analogy is also complete if the rotation axis is fixed by external constraints.)

Evident examples of a spherical top are a uniform sphere and a uniform spherical shell; a less obvious example is a uniform cube – with masses either concentrated in vertices, or uniformly spread over the faces, or uniformly distributed over the volume. Again, in this case any axis passing through the center of mass is principal and has the same principal moment of inertia. For a sphere, this is natural; for a cube, rather surprising – but may be confirmed by a direct calculation.

### 4.3. Fixed-axis rotation

Now we are well equipped for a discussion of rigid body’s rotational dynamics. The general equation of this dynamics is given by Eq. (1.38), which is valid for dynamics of any system of particles – either rigidly connected or not:
\[
\dot{\mathbf{L}} = \mathbf{\tau}, \quad (4.33)
\]
where \( \mathbf{\tau} \) is the net torque of external forces. Let us start exploring this equation from the simplest case when the axis of rotation, i.e. the direction of vector \( \omega \), is fixed by some external constraints. Directing
the $z$-axis along this vector, we have $\omega_x = \omega_y = 0$. According to Eq. (22), in this case, the $z$-component of the angular momentum,

$$ L_z = I_{zz} \omega_z, $$

where $I_{zz}$, though not necessarily one of the principal moments of inertia, still may be calculated using Eq. (24):

$$ I_{zz} = \sum m \rho_z^2 = \sum m(x^2 + y^2), $$

with $\rho_z$ being the distance of each particle from the rotation axis $z$. According to Eq. (15), in this case the rotational kinetic energy is just

$$ T_{\text{rot}} = \frac{I_{zz} \omega_z^2}{2}. $$

Moreover, it is straightforward to show that if the rotation axis is fixed, Eqs. (34)-(36) are valid even if the axis does not pass through the center of mass – provided that the distances $\rho_z$ are now measured from that axis. (The proof is left for the reader’s exercise.)

As a result, we may not care about other components of the vector $L$, and use just one component of Eq. (33),

$$ \dot{L}_z = \tau_z, $$

because it, when combined with Eq. (34), completely determines the dynamics of rotation:

$$ I_{zz} \dot{\omega}_z = \tau_z, \quad \text{i.e.} \quad I_{zz} \dot{\theta}_z = \tau_z, $$

where $\theta_z$ is the angle of rotation about the axis, so that $\omega_z = \dot{\theta}_z$. The scalar relations (34), (36) and (38), describing rotation about a fixed axis, are completely similar to the corresponding formulas of 1D motion of a single particle, with $\omega_z$ corresponding to the usual (“linear”) velocity, the angular momentum component $L_z$ – to the linear momentum, and $I_z$ – to particle’s mass.

The resulting motion about the axis is also frequently similar to that of a single particle. As a simple example, let us consider what is called the physical pendulum (Fig. 5) – a rigid body free to rotate about a fixed horizontal axis that does not pass through the center of mass 0, in a uniform gravity field $g$.

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8 Note that according to Eq. (22), other Cartesian components of the angular momentum, $L_x$ and $L_y$, may be different from zero, and may even evolve in time. The corresponding torques $\tau_x$ and $\tau_y$, which obey Eq. (33), are automatically provided by the external forces that keep the rotation axis fixed.
Let us drop the perpendicular from point 0 to the rotation axis, and call the oppositely directed vector \( l \) – see the dashed arrow in Fig. 5. Then the torque (relative to the rotation axis 0’) of the forces keeping the axis fixed is zero, and the only contribution to the net torque is due to gravity alone:

\[
\tau_{\text{in \ 0}} = \sum r_{\text{in \ 0}} \times F = \sum (l + r_{\text{in \ 0}}) \times mg = \sum m(l \times g) + \sum m r_{\text{in \ 0}} \times g = Ml \times g. \tag{4.39}
\]

(The last step used the facts that point 0 is the center of mass, so that the second term in the right-hand side equals zero, and that the vectors \( l \) and \( g \) are the same for all particles of the body.)

This result shows that the torque is directed along the rotation axis, and its (only) component \( \tau_z \) is equal to \(-Mgl\sin\theta\), where \( \theta \) is the angle between the vectors \( l \) and \( g \), i.e. the angular deviation of the pendulum from the position of equilibrium – see Fig. 5 again. As a result, Eq. (38) takes the form,

\[
I' \dot{\theta} = -Mgl \sin \theta, \tag{4.40}
\]

where \( I' \) is the moment of inertia for rotation about the axis 0’ rather than about the center of mass. This equation is identical to Eq. (1.18) for the point-mass (sometimes called “mathematical”) pendulum, with the small-oscillation frequency

\[
\Omega = \left( \frac{Mgl}{l'} \right)^{1/2}. \tag{4.41}
\]

As a sanity check, in the simplest case when the linear size of the body is much smaller than the suspension length \( l \), Eq. (35) yields \( I' = Ml^2 \), and Eq. (41) reduces to the well-familiar formula \( \Omega = (g/l)^{1/2} \) for the point-mass pendulum.

Now let us discuss the situations when a rigid body not only rotates but also moves as a whole. As we already know from our introductory chapter, the total linear momentum of the body,

\[
P = \sum m \mathbf{v} = \sum m \mathbf{\dot{r}} = \frac{d}{dt} \sum m \mathbf{r}, \tag{4.42}
\]

satisfies the 2\(^{\text{nd}}\) Newton law in the form (1.30). Using the definition (13) of the center of mass, the momentum may be represented as

\[
P = M \dot{\mathbf{R}} = M \mathbf{V}, \tag{4.43}
\]

so Eq. (1.30) may be rewritten as

\[
M \ddot{\mathbf{V}} = \mathbf{F}, \tag{4.44}
\]

where \( \mathbf{F} \) is the vector sum of all external forces. This equation shows that the center of mass of the body moves exactly like a point particle of mass \( M \), under the effect of the net force \( \mathbf{F} \). In many cases, this fact makes the translational dynamics of a rigid body absolutely similar to that of a point particle.

The situation becomes more complex if some of the forces contributing to the vector sum \( \mathbf{F} \) depend on the rotation of the same body, i.e. if its rotational and translational motions are coupled. Analysis of such coupled motion is rather straightforward if the direction of the rotation axis does not change in time, and hence Eqs. (35)-(36) are still valid. Possibly the simplest example is a round cylinder (say, a wheel) rolling on a surface without slippage (Fig. 6). Here the no-slippage condition may be represented as the requirement of the net velocity of the particular wheel’s point A that touches the surface to equal zero – in the reference frame connected to the surface. For the simplest case of plane surface (Fig. 6a), this condition may be spelled out using Eq. (10), giving the following relation between the angular velocity \( \omega \) of the wheel and the linear velocity \( V \) of its center:
Such kinematic relations are essentially holonomic constraints, which reduce the number of degrees of freedom of the system. For example, without the no-slippage condition (45), the wheel on a plane surface has to be considered as a system with two degrees of freedom, making its total kinetic energy (14) a function of two independent generalized velocities, say $V$ and $\omega$:

$$T = T_{\text{rot}} = \frac{M}{2} V^2 + \frac{I}{2} \omega^2.$$  

(4.46)

Using Eq. (45) we may eliminate, for example, the linear velocity and reduce Eq. (46) to

$$T = \frac{M}{2} (\omega r)^2 + \frac{I}{2} \omega^2 \equiv \frac{I_{\text{ef}}}{2} \omega^2, \quad \text{where} \quad I_{\text{ef}} \equiv I + Mr^2.$$  

(4.47)

This result may be interpreted as the kinetic energy of pure rotation of the wheel about the instantaneous rotation axis $A$, with $I_{\text{ef}}$ being the moment of inertia about that axis, satisfying Eq. (29).

Kinematic relations are not always as simple as Eq. (45). For example, if a wheel is rolling on a concave surface (Fig. 6b), we need to relate the angular velocities of the wheel’s rotation about its axis 0’ (say, $\omega$) and that (say, $\Omega$) of its axis’ rotation about the center 0 of curvature of the surface. A popular error here is to write $\Omega = -(r/R) \omega$ [WRONG!]. A prudent way to derive the correct relation is to note that Eq. (45) holds for this situation as well, and on the other hand, the same linear velocity of the wheel’s center may be expressed as $V = (R - r) \Omega$. Combining these formulas, we get the correct relation

$$\Omega = - \frac{r}{R - r} \omega.$$  

(4.48)

Another famous example of the relation between the translational and rotational motion is given by the “sliding ladder” problem (Fig. 7). Let us analyze it for the simplest case of negligible friction, and the ladder’s thickness small in comparison with its length $l$. 

$$ \mathbf{R} = \{X, Y\} $$

Fig. 4.7. The sliding ladder problem.
To use the Lagrangian formalism, we may write the kinetic energy of the ladder as the sum (14) of its translational and rotational parts:

$$T = \frac{M}{2} \left( \dot{X}^2 + \dot{Y}^2 \right) + \frac{I}{2} \dot{\alpha}^2, \quad (4.49)$$

where $X$ and $Y$ are the Cartesian coordinates of its center of mass in an inertial reference frame, and $I$ is the moment of inertia for rotation about the $z$-axis passing through the center of mass. (For the uniformly distributed mass, an elementary integration of Eq. (35) yields $I = Ml^2/12$.) In the reference frame with the center in the corner 0, both $X$ and $Y$ may be simply expressed via the angle $\alpha$:

$$X = \frac{l}{2} \cos \alpha, \quad Y = \frac{l}{2} \sin \alpha. \quad (4.50)$$

(The easiest way to obtain these relations is to notice that the dashed line in Fig. 7 has length $l/2$, and the same slope $\alpha$ as the ladder.) Plugging these expressions into Eq. (49), we get

$$T = \frac{I_{ef}}{2} \dot{\alpha}^2, \quad I_{ef} \equiv I + M \left( \frac{l}{2} \right)^2 = \frac{1}{3} Ml^2. \quad (4.51)$$

Since the potential energy of the ladder in the gravity field may be also expressed via the same angle,

$$U = MgY = Mg \frac{l}{2} \sin \alpha, \quad (4.52)$$

$\alpha$ may be conveniently used as the (only) generalized coordinate of the system. Even without writing the Lagrange equation of motion for that coordinate, we may notice that since the Lagrangian function $L \equiv T - U$ does not depend on time explicitly, and the kinetic energy (51) is a quadratic-homogeneous function of the generalized velocity $\dot{\alpha}$, the full mechanical energy,

$$E \equiv T + U = \frac{I_{ef}}{2} \dot{\alpha}^2 + Mg \frac{l}{2} \sin \alpha = \frac{Mgl}{2} \left( \frac{1}{3g} \dot{\alpha}^2 + \sin \alpha \right), \quad (4.53)$$

is conserved, giving us the first integral of motion. Moreover, Eq. (53) shows that the system’s energy (and hence dynamics) is identical to that of a physical pendulum with an unstable fixed point $\alpha_1 = \pi/2$, a stable fixed point at $\alpha_2 = -\pi/2$, and frequency

$$\Omega = \left( \frac{3g}{2l} \right)^{1/2} \quad (4.54)$$

of small oscillations near the latter point. (Of course, this fixed point cannot be reached in the simple geometry shown in Fig. 7, where the ladder’s fall on the floor would change its equations of motion. Moreover, even before that, the left end of the ladder may detach from the wall. The analysis of this issue is left for the reader’s exercise.)

4.4. Free rotation

Now let us proceed to the more complex case when the rotation axis is not fixed. A good illustration of the complexity arising in this case comes from the case of a rigid body left alone, i.e. not subjected to external forces and hence with its potential energy $U$ constant. Since in this case, according
to Eq. (44), the center of mass moves (as measured from any inertial reference frame) with a constant velocity, we can always use a convenient inertial reference frame with the origin at that point. From the point of view of such a frame, the body’s motion is a pure rotation, and $T_{\text{tran}} = 0$. Hence, the system’s Lagrangian equals just the rotational energy (15), which is, first, a quadratic-homogeneous function of the components $\omega_j$ (which may be taken for generalized velocities), and, second, does not depend on time explicitly. As we know from Chapter 2, in this case the mechanical energy, here equal to $T_{\text{rot}}$ alone, is conserved. According to Eq. (15), for the principal-axes components of the vector $\mathbf{\omega}$, this means

$$T_{\text{rot}} = \sum_{j=1}^{3} \frac{I_j}{2} \omega_j^2 = \text{const}.$$  \hspace{1cm} (4.55)

Next, as Eq. (33) shows, in the absence of external forces, the angular momentum $\mathbf{L}$ of the body is conserved as well. However, though we can certainly use Eq. (26) to represent this fact as

$$\mathbf{L} = \sum_{j=1}^{3} I_j \omega_j \mathbf{n}_j = \text{const},$$  \hspace{1cm} (4.56)

where $\mathbf{n}_j$ are the principal axes, this does not mean that all components $\omega_j$ are constant, because the principal axes are fixed relative to the rigid body, and hence may rotate with it.

Before exploring these complications, let us briefly mention two conceptually trivial, but practically very important, particular cases. The first is a spherical top ($I_1 = I_2 = I_3 = I$). In this case, Eqs. (55) and (56) imply that all components of the vector $\mathbf{\omega} = \mathbf{L}/I$, i.e. both the magnitude and the direction of the angular velocity are conserved, for any initial spin. In other words, the body conserves its rotation speed and axis direction, as measured in an inertial frame. The most obvious example is a spherical planet. For example, our Mother Earth, rotating about its axis with angular velocity $\omega = 2\pi(1 \text{ day}) \approx 7.3 \times 10^{-5} \text{ s}^{-1}$, keeps its axis at a nearly constant angle of 23°27' to the ecliptic pole, i.e. the axis normal to the plane of its motion around the Sun. (In Sec. 6 below, we will discuss some very slow motions of this axis, due to gravity effects.)

Spherical tops are also used in the most accurate gyroscopes, usually with gas-jet or magnetic suspension in vacuum. If done carefully, such systems may have spectacular stability. For example, the gyroscope system of the Gravity Probe B satellite experiment, flown in 2004-5, was based on quartz spheres – round with precision of about 10 nm and covered with superconducting thin films (which enabled their magnetic suspension and monitoring). The whole system was stable enough to measure that the so-called geodetic effect in general relativity (essentially, the space curving by Earth’s mass), resulting in the axis’ precession by only 6.6 arc seconds per year, i.e. with a precession frequency of just ~$10^{-11} \text{ s}^{-1}$, agrees with theory with a record ~0.3% accuracy.\footnote{Still, the main goal of this rather expensive (~$750M) project, an accurate measurement of a more subtle relativistic effect, the so-called frame-dragging drift (also called “the Schiff precession”), predicted to be about 0.04 arc seconds per year, has not been achieved.}

The second simple case is that of the symmetric top ($I_1 = I_2 \neq I_3$), with the initial vector $\mathbf{L}$ aligned with the main principal axis. In this case, $\mathbf{\omega} = \mathbf{L}/I_3 = \text{const}$, so that the rotation axis is conserved.\footnote{This is also true for an asymmetric top, i.e. an arbitrary body (with, say, $I_1 < I_2 < I_3$), but in this case the alignment of the vector $\mathbf{L}$ with the axis $\mathbf{n}_2$ corresponding to the intermediate moment of inertia, is unstable.} Such tops, typically in the shape of a flywheel (heavy, flat rotor), and supported by a three-ring gimbal system...
(which allows for torque-free rotation in all three directions),\(^{11}\) are broadly used in more common
gyroscopes – core parts of automatic guidance systems, for example, in ships, airplanes, missiles, etc. Even if the ship’s hull wobbles and/or drifts, the suspended gyroscope sustains its direction relative to the Earth – which is a sufficiently inertial reference frame for these applications.\(^{12}\)

However, in the general case with no such special initial alignment, the dynamics of symmetric
tops is more complicated. In this case, the vector \(\mathbf{L}\) is still conserved, including its direction, but the vector \(\mathbf{\omega}\) is not. Indeed, let us direct the \(\mathbf{n}_2\) axis normally to the common plane of vectors \(\mathbf{L}\) and the current instantaneous direction \(\mathbf{n}_3\) of the main principal axis (in Fig. 8 below, the plane of the drawing); then, in that particular instant, \(L_2 = 0\). Now let us recall that in a symmetric top, the axis \(\mathbf{n}_2\) is a principal one. According to Eq. (26) with \(j = 2\), the corresponding component \(\omega_2\) has to be equal to \(L_2/L_2\), so it is equal to zero. This means that the vector \(\mathbf{\omega}\) lies in this plane (the common plane of vectors \(\mathbf{L}\) and \(\mathbf{n}_3\)) as well – see Fig. 8a.

![Figure 4.8](image)

Now consider any point located on the main principal axis \(\mathbf{n}_3\), and hence on the plane \([\mathbf{n}_3, \mathbf{L}]\). Since \(\mathbf{\omega}\) is the instantaneous axis of rotation, according to Eq. (9), the instantaneous velocity \(\mathbf{v} = \mathbf{\omega} \times \mathbf{r}\) of the point is directed normally to that plane. Since this is true for each point of the main axis (besides only one, with \(\mathbf{r} = 0\), i.e. the center of mass, which does not move), this axis as a whole has to move perpendicular to the common plane of the vectors \(\mathbf{L}, \mathbf{\omega}, \mathbf{n}_3\). Since this conclusion is valid for any moment of time, it means that the vectors \(\mathbf{\omega}\) and \(\mathbf{n}_3\) rotate about the space-fixed vector \(\mathbf{L}\) together, with some angular velocity \(\omega_{\text{pre}}\), at each moment staying within one plane. This effect is usually called the free precession (or “torque-free”, or “regular”) precession, and has to be clearly distinguished it from the completely different effect of the torque-induced precession, which will be discussed in the next section.

To calculate \(\omega_{\text{pre}}\), let us represent the instant vector \(\mathbf{\omega}\) as a sum of not its Cartesian components (as in Fig. 8a), but rather of two non-orthogonal vectors directed along \(\mathbf{n}_3\) and \(\mathbf{L}\) (Fig. 8b):

\[
\mathbf{\omega} = \omega_{\text{rot}} \mathbf{n}_3 + \omega_{\text{pre}} \mathbf{n}_L, \quad \mathbf{n}_L \equiv \frac{\mathbf{L}}{L}.
\]  

\(^{11}\) See, for example, a very nice animation available online at [http://en.wikipedia.org/wiki/Gimbal](http://en.wikipedia.org/wiki/Gimbal).

\(^{12}\) Much more compact (and much less accurate) gyroscopes used, for example, in smartphones and tablet computers, are based on a more subtle effect of rotation on mechanical oscillator’s frequency, and are implemented as micro-electromechanical systems (MEMS) on silicon chip surfaces – see, e.g., Chapter 22 in V. Kaajakari, *Practical MEMS*, Small Gear Publishing, 2009. Too bad I have no time/space to cover their theory!
Fig. 8b shows that $\omega_{\text{rot}}$ has the meaning of the angular velocity of rotation of the body about its main principal axis, while $\omega_{\text{pre}}$ is the angular velocity of rotation of that axis about the constant direction of the vector $\mathbf{L}$, i.e. the frequency of precession, i.e. exactly what we are trying to find. Now $\omega_{\text{pre}}$ may be readily calculated from the comparison of two panels of Fig. 8, by noticing that the same angle $\theta$ between the vectors $\mathbf{L}$ and $\mathbf{n}_3$ participates in two relations:

$$\sin \theta = \frac{L_1}{L} = \frac{\omega_1}{\omega_{\text{pre}}}.$$  \hspace{1cm} (4.58)

Since the $\mathbf{n}_1$-axis is a principal one, we may use Eq. (26) for $j = 1$, i.e. $L_1 = I_1 \omega_1$, to eliminate $\omega_1$ from Eq. (58), and get a very simple formula

$$\omega_{\text{pre}} = \frac{L}{I_1}. \hspace{1cm} (4.59)$$

This result shows that the precession frequency is constant and independent of the alignment of the vector $\mathbf{L}$ with the main principal axis $\mathbf{n}_3$, while the amplitude of this motion (characterized by the angle $\theta$) does depend on the alignment, and vanishes if $\mathbf{L}$ is parallel to $\mathbf{n}_3$. Note also that if all principal moments of inertia are of the same order, $\omega_{\text{pre}}$ is of the same order as the total angular speed $\omega \equiv |\omega|$ of rotation.

Now let us briefly discuss the free precession in the general case of an “asymmetric top”, i.e. a body with arbitrary $I_1 \neq I_2 \neq I_3$. In this case, the effect is more complex because here not only the direction but also the magnitude of the instantaneous angular velocity $\omega$ may evolve in time. If we are only interested in the relation between the instantaneous values of $\omega_j$ and $L_j$, i.e. the “trajectories” of the vectors $\omega$ and $\mathbf{L}$ as observed from the reference frame $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ of the principal axes of the body (rather than the explicit law of their time evolution), they may be found directly from the conservation laws. (Let me emphasize again that the vector $\mathbf{L}$, being constant in an inertial reference frame, generally evolves in the frame rotating with the body.) Indeed, Eq. (55) may be understood as the equation of an ellipsoid in Cartesian coordinates $\{\omega_1, \omega_2, \omega_3\}$, so that for a free body, the vector $\omega$ has to stay on the surface of that ellipsoid. On the other hand, since the reference frame rotation preserves the length of any vector, the magnitude (but not the direction!) of the vector $\mathbf{L}$ is also an integral of motion in the moving frame, and we can write

$$L^2 \equiv \sum_{j=1}^{3} L_j^2 = \sum_{j=1}^{3} I_j \omega_j^2 = \text{const}. \hspace{1cm} (4.60)$$

Hence the trajectory of the vector $\omega$ follows the closed curve formed by the intersection of two ellipsoids, (55) and (60). It is evident that this trajectory is generally “taco-edge-shaped”, i.e. more complex than a planar circle, but never very complex either.

13 For our Earth, the free precession amplitude is so small (corresponding to sub-10-m linear displacements of the Earth surface) that this effect is of the same order as other, more irregular motions of the rotation axis, resulting from the turbulent fluid flow effects in planet’s interior and its atmosphere.

14 It is frequently called the Poinset’s ellipsoid, named after Louis Poinset (1777-1859) who has made several important contributions to rigid body mechanics.

15 Curiously, the “wobbling” motion along such trajectories was observed not only for macroscopic rigid bodies, but also for heavy atomic nuclei – see, e.g., N. Sensharma et al., Phys. Rev. Lett. 124, 052501 (2020).
The same argument may be repeated for the vector $L$, for whom the first form of Eq. (60) describes a sphere, and Eq. (55), another ellipsoid:

$$T_{\text{rot}} = \frac{1}{2} \sum_{j=1}^{3} I_j L_j^2 = \text{const}. \quad (4.61)$$

On the other hand, if we are interested in the trajectory of the vector $\omega$ as observed from an inertial frame (in which the vector $L$ stays still), we may note that the general relation (15) for the same rotational energy $T_{\text{rot}}$ may also be rewritten as

$$T_{\text{rot}} = \frac{1}{2} \sum_{j=1}^{3} \omega_j \sum_{j'=1}^{3} I_{jj'} \omega_{j'}. \quad (4.62)$$

But according to the Eq. (22), the second sum on the right-hand side is nothing more than $L_j$, so that

$$T_{\text{rot}} = \frac{1}{2} \sum_{j=1}^{3} \omega_j L_j = \frac{1}{2} \omega \cdot L. \quad (4.63)$$

This equation shows that for a free body ($T_{\text{rot}} = \text{const}, L = \text{const}$), even if the vector $\omega$ changes in time, its endpoint should stay within a plane perpendicular to the angular momentum $L$. (Earlier, we have seen that for the particular case of the symmetric top – see Fig. 8b, but for an asymmetric top, the trajectory of the endpoint may not be circular.)

If we are interested not only in the trajectory of the vector $\omega$, but also the law of its evolution in time, it may be calculated using the general Eq. (33) expressed in the principal components $\omega_j$. For that, we have to recall that Eq. (33) is only valid in an inertial reference frame, while the frame $\{n_1, n_2, n_3\}$ may rotate with the body and hence is generally not inertial. We may handle this problem by applying, to the vector $L$, the general kinematic relation (8):

$$\frac{dL}{dt} \bigg|_{\text{lab}} = \frac{dL}{dt} \bigg|_{\text{move}} + \omega \times L. \quad (4.64)$$

Combining it with Eq. (33), in the moving frame we get

$$\frac{dL}{dt} + \omega \times L = \tau, \quad (4.65)$$

where $\tau$ is the external torque. In particular, for the principal-axis components $L_j$, related to the components $\omega_j$ by Eq. (26), the vector equation (65) is reduced to a set of three scalar Euler equations

$$I_j \omega_j + (I_{jj'} - I_{j'}) \omega_j \omega_{j'} = \tau_j, \quad (4.66)$$

where the set of indices $\{j, j', j''\}$ has to follow the usual “right” order – e.g., $\{1, 2, 3\}$, etc.\(^{16}\)

In order to get a feeling how do the Euler equations work, let us return to the particular case of a free symmetric top ($\tau_1 = \tau_2 = \tau_3 = 0, I_1 = I_2 \neq I_3$). In this case, $I_1 - I_2 = 0$, so that Eq. (66) with $j = 3$ yields $\omega_3 = \text{const}$, while the equations for $j = 1$ and $j = 2$ take the following simple form:

\(^{16}\) These equations are of course valid in the simplest case of the fixed rotation axis as well. For example, if $\omega = n_2 \omega_2$, i.e. $\omega_z = \omega_2 = 0$, Eq. (66) is reduced to Eq. (38).
\[ \dot{\omega}_1 = -\Omega_{\text{pre}} \omega_2, \quad \dot{\omega}_2 = \Omega_{\text{pre}} \omega_1, \]  
(4.67)

where \( \Omega_{\text{pre}} \) is a constant determined by both the system parameters and the initial conditions:

\[ \Omega_{\text{pre}} \equiv \omega_3 \frac{I_3 - I_1}{I_1}. \]  
(4.68)

The system of two equations (67) has a sinusoidal solution with frequency \( \Omega_{\text{pre}} \), and describes a uniform rotation of the vector \( \omega \), with that frequency, about the main axis \( n_3 \). This is just another representation of the torque-free precession analyzed above, this time as observed from the rotating body. Evidently, \( \Omega_{\text{pre}} \) is substantially different from the frequency \( \omega_{\text{pre}} \) (59) of the precession as observed from the lab frame; for example, \( \Omega_{\text{pre}} \) vanishes for the spherical top (with \( I_1 = I_2 = I_3 \)), while \( \omega_{\text{pre}} \), in this case, is equal to the rotation frequency.

Unfortunately, for the rotation of an asymmetric top (i.e., an arbitrary rigid body), when no component \( \omega_j \) is conserved, the Euler equations (66) are strongly nonlinear even in the absence of any external torque, and a discussion of their solutions would take more time than I can afford.  

4.5. Torque-induced precession

The dynamics of rotation becomes even more complex in the presence of external forces. Let us consider the most important and counter-intuitive effect of torque-induced precession, for the simplest case of an axially-symmetric body (which is a particular case of the symmetric top, \( I_1 = I_2 \neq I_3 \)), supported at some point \( A \) of its symmetry axis, that does not coincide with the center of mass \( 0 \) – see Fig. 9.

![Fig. 4.9. Symmetric top in the gravity field: (a) a side view at the system and (b) the top view at the evolution of the horizontal component of the angular momentum vector.](image)

The uniform gravity field \( g \) creates bulk-distributed forces that, as we know from the analysis of the physical pendulum in Sec. 3, are equivalent to a single force \( Mg \) applied in the center of mass – in Fig. 9, point \( 0 \). The torque of this force relative to the support point \( A \) is

\[ \tau = r_0 |_{\text{in } A} \times Mg = Ml n_3 \times g. \]  
(4.69)

Hence the general equation (33) of the angular momentum evolution (valid in any inertial frame, for example the one with an origin in point A) becomes

\[ \mathbf{L} = Ml \mathbf{n}_3 \times \mathbf{g}. \]  \hspace{1cm} (4.70)

Despite the apparent simplicity of this (exact!) equation, its analysis is straightforward only in the limit when the top is launched spinning about its symmetry axis \( \mathbf{n}_3 \) with a very high angular velocity \( \omega_{\text{rot}} \). In this case, we may neglect the contribution to \( \mathbf{L} \) due to a relatively small precession velocity \( \omega_{\text{pre}} \) (still to be calculated), and use Eq. (26) to write

\[ \mathbf{L} = I_3 \omega = I_3 \omega_{\text{rot}} \mathbf{n}_3. \]  \hspace{1cm} (4.71)

Then Eq. (70) shows that the vector \( \dot{\mathbf{L}} \) is perpendicular to both \( \mathbf{n}_3 \) (and hence \( \mathbf{L} \)) and \( \mathbf{g} \), i.e. lies within the horizontal plane and is perpendicular to the horizontal component \( \mathbf{L}_{xy} \) of the vector \( \mathbf{L} \) – see Fig. 9b. Since, according to Eq. (70), the magnitude of this vector is constant, \( | \mathbf{L} | = mgl \sin \theta \), the vector \( \mathbf{L} \) (and hence the body’s main axis) rotates about the vertical axis with the following angular velocity:

\[ \omega_{\text{pre}} = \frac{L}{L_{xy}} = \frac{Mgl \sin \theta}{L \sin \theta} \equiv \frac{Mgl}{L} = \frac{Mgl}{I_3 \omega_{\text{rot}}}. \]  \hspace{1cm} (4.72)

Thus, very counter-intuitively, the fast-rotating top does not follow the external, vertical force and, in addition to fast spinning about the symmetry axis \( \mathbf{n}_3 \), performs a revolution, called the \textit{torque-induced precession}, about the vertical axis. Note that, similarly to the free-precession frequency (59), the torque-induced precession frequency (72) does not depend on the initial (and sustained) angle \( \theta \). However, the torque-induced precession frequency is inversely (rather than directly) proportional to \( \omega_{\text{rot}} \). This fact makes the above simple theory valid in many practical cases. Indeed, Eq. (71) is quantitatively valid if the contribution of the precession into \( \mathbf{L} \) is relatively small: \( I \omega_{\text{pre}} \ll I_3 \omega_{\text{rot}} \), where \( I \) is a certain effective moment of inertia for the precession - to be calculated below. Using Eq. (72), this condition may be rewritten as

\[ \omega_{\text{rot}} >> \left( \frac{Mgl}{I_3^2} \right)^{1/2}. \]  \hspace{1cm} (4.73)

According to Eq. (16), for a body of not too extreme proportions, i.e. with all linear dimensions of the order of the same length scale \( l \), all inertia moments are of the order of \( Ml^2 \), so that the right-hand side of Eq. (73) is of the order of \( (g/l)^{1/2} \), i.e. comparable with the frequency of small oscillations of the same body as the physical pendulum, i.e. at the absence of its fast rotation.

To develop a qualitative theory that would be valid beyond such approximate treatment, the Euler equations (66) may be used, but are not very convenient. A better approach, suggested by the same L. Euler, is to introduce a set of three independent angles between the principal axes \( \{ \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 \} \) bound to the rigid body, and the axes \( \{ \mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z \} \) of an inertial reference frame (Fig. 10), and then express the basic equation (33) of rotation, via these angles. There are several possible options for the definition of such angles; Fig. 10 shows the set of \textit{Euler angles}, most convenient for analyses of fast rotation.\(^\text{18}\) As one can see, the first Euler angle, \( \theta \), is the usual polar angle measured from the \( \mathbf{n}_z \)-axis to

\(^{18}\) Of the several choices more convenient in the absence of fast rotation, the most common is the set of so-called \textit{Tait-Brian angles} (called the \textit{yaw}, \textit{pitch}, and \textit{roll}), which are broadly used for aircraft and maritime navigation.
the $n_3$-axis. The second one is the azimuthal angle $\phi$, measured from the $n_x$-axis to the so-called line of nodes formed by the intersection of planes $[n_x, n_y]$ and $[n_1, n_2]$. The last Euler angle, $\psi$, is measured within the plane $[n_1, n_2]$, from the line of nodes to axis $n_1$-axis. For example, in the simple picture of slow force-induced precession of a symmetric top, that was discussed above, the angle $\theta$ is constant, the angle $\psi$ changes rapidly, with the rotation velocity $\omega_{\text{rot}}$, while the angle $\phi$ evolves with the precession frequency $\omega_{\text{pre}}$ (72).

Now we can express the principal-axes components of the instantaneous angular velocity vector, $\omega_1$, $\omega_2$, and $\omega_3$, as measured in the lab reference frame, in terms of the Euler angles. This may be readily done by calculating, from Fig. 10, the contributions of the Euler angles’ evolution to the rotation about each principal axis, and then adding them up:

$$
\begin{align*}
\omega_1 &= \phi \sin \theta \sin \psi + \dot{\theta} \cos \psi, \\
\omega_2 &= \phi \sin \theta \cos \psi - \dot{\theta} \sin \psi, \\
\omega_3 &= \phi \cos \theta + \psi.
\end{align*}
$$

These relations enable the expression of the kinetic energy of rotation (25) and the angular momentum components (26) via the generalized coordinates $\theta$, $\phi$, and $\psi$ and their time derivatives (i.e. the corresponding generalized velocities), and then using the powerful Lagrangian formalism to derive their equations of motion. This is especially simple to do in the case of symmetric tops (with $I_1 = I_2$), because plugging Eqs. (74) into Eq. (25) we get an expression,

$$
T_{\text{rot}} = \frac{I_1}{2} (\dot{\theta}^2 + \phi^2 \sin^2 \theta) + \frac{I_1}{2} (\dot{\phi} \cos \theta + \psi)^2, 
$$

which does not include explicitly either $\phi$ or $\psi$. (This reflects the fact that for a symmetric top we can always select the $n_1$-axis to coincide with the line of nodes, and hence take $\psi = 0$ at the considered moment of time. Note that this trick does not mean we can take $\psi = 0$, because the $n_1$-axis, as observed from an inertial reference frame, moves!) Now we should not forget that at the torque-induced precession, the center of mass moves as well (see, e.g., Fig. 9), so that according to Eq. (14), the total kinetic energy of the body is the sum of two terms,

$$
T = T_{\text{rot}} + T_{\text{trans}}, \\
T_{\text{trans}} = \frac{M}{2} V^2 = \frac{M}{2} I_2 (\dot{\theta}^2 + \phi^2 \sin^2 \theta),
$$
While its potential energy is just
\[ U = Mgl \cos \theta + \text{const}. \]  

(4.77)

Now we could readily write the Lagrange equations of motion for the Euler angles, but it is simpler to immediately notice that according to Eqs. (75)-(77), the Lagrangian function, \( T - U \), does not depend explicitly on the “cyclic” coordinates \( \varphi \) and \( \psi \), so that the corresponding generalized momenta (2.31) are conserved:
\[ p_\varphi \equiv \frac{\partial T}{\partial \dot{\varphi}} = I_A \dot{\varphi} \sin^2 \theta + I_3 (\dot{\varphi} \cos \theta + \dot{\psi}) \cos \theta = \text{const}, \]  

(4.78)

\[ p_\psi \equiv \frac{\partial T}{\partial \dot{\psi}} = I_3 (\dot{\varphi} \cos \theta + \dot{\psi}) = \text{const}, \]  

(4.79)

where \( I_A \equiv I_1 + Ml^2 \). (According to Eq. (29), \( I_A \) is just the body’s moment of inertia for rotation about a horizontal axis passing through the support point A.) According to the last of Eqs. (74), \( p_\varphi \) is just \( L_3 \), i.e. the angular momentum’s component along the precessing axis \( n_3 \). On the other hand, by its very definition (78), \( p_\psi \) is \( L_z \), i.e. the same vector \( L \)’s component along the static axis \( z \). (Actually, we could foresee in advance the conservation of both these components of \( L \) for our system, because the vector (69) of the external torque is perpendicular to both \( n_3 \) and \( n_z \).) Using this notation, and solving the simple system of linear equations (78)-(79) for the angle derivatives, we get
\[ \dot{\varphi} = \frac{L_z - L_3 \cos \theta}{I_A \sin^2 \theta}, \quad \dot{\psi} = \frac{L_3}{I_3} - \frac{L_z - L_3 \cos \theta}{I_A \sin^2 \theta} \cos \theta. \]  

(4.80)

One more conserved quantity in this problem is the full mechanical energy19
\[ E \equiv T + U = \frac{I_A}{2} (\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \frac{I_3}{2} (\dot{\varphi} \cos \theta + \dot{\psi})^2 + Mgl \cos \theta. \]  

(4.81)

Plugging Eqs. (80) into Eq. (81), we get a first-order differential equation for the angle \( \theta \), which may be represented in the following physically transparent form:
\[ \frac{I_A}{2} \dot{\theta}^2 + U_{\text{ef}}(\theta) = E, \quad U_{\text{ef}}(\theta) = \frac{(L_z - L_3 \cos \theta)^2}{2I_A \sin^2 \theta} + \frac{L_3^2}{2I_3} + Mgl \cos \theta + \text{const}. \]  

(4.82)

Thus, similarly to the planetary problems considered in Sec. 3.4, the torque-induced precession of a symmetric top has been reduced (without any approximations!) to a 1D problem of the motion of one of its degrees of freedom, the polar angle \( \theta \), in the effective potential \( U_{\text{ef}}(\theta) \). According to Eq. (82), very similar to Eq. (3.44) for the planetary problem, this potential is the sum of the actual potential energy \( U \) given by Eq. (77), and a contribution from the kinetic energy of motion along two other angles. In the absence of rotation about the axes \( n_2 \) and \( n_3 \) (i.e., \( L_z = L_3 = 0 \)), Eq. (82) is reduced to the first integral of the equation (40) of motion of a physical pendulum, with \( I' = I_A \). If the rotation is present, then (besides the case of very special initial conditions when \( \theta(0) = 0 \) and \( L_z = L_3 \))20 the first

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19 Indeed, since the Lagrangian does not depend on time explicitly, \( H = \text{const} \), and since the full kinetic energy \( T \) (75)-(76) is a quadratic-homogeneous function of the generalized velocities, \( E = H \).

20 In that simple case, the body continues to rotate about the vertical symmetry axis: \( \dot{\theta}(t) = 0 \). Note, however, that such motion is stable only if the spinning speed is sufficiently high – see Eq. (85) below.
contribution to $U_{\text{ef}}(\theta)$ diverges at $\theta \to 0$ and $\pi$, so that the effective potential energy has a minimum at some non-zero value $\theta_0$ of the polar angle $\theta$ – see Fig. 11.

Fig. 4.11. The effective potential energy $U_{\text{ef}}$ of the symmetric top, given by Eq. (82), as a function of the polar angle $\theta$, for a particular value (0.95) of the ratio $r \equiv L_z/L_3$ (so that at $\omega_\text{rot} > \omega_\text{th}$, $\theta_0 = \cos^{-1}r \approx 0.1011\pi$), and several values of the ratio $\omega_\text{rot}/\omega_\text{th}$. 

If the initial angle $\theta(0)$ is equal to this value $\theta_0$, i.e. if the initial effective energy is equal to its minimum value $U_{\text{ef}}(\theta_0)$, the polar angle remains constant through the motion: $\theta(t) = \theta_0$. This corresponds to the pure torque-induced precession whose angular velocity is given by the first of Eqs. (80):

$$\omega_{\text{pre}} \equiv \phi = \frac{L_z - L_3 \cos \theta_0}{I_A \sin^2 \theta_0}. \quad (4.83)$$

The condition for finding $\theta_0$, $dU_{\text{ef}}/d\theta = 0$, is a transcendental algebraic equation that cannot be solved analytically for arbitrary parameters. Indeed, in this limit the $Mgl$-proportional contribution to $U_{\text{ef}}$ is small, and we may analyze its effect by successive approximations. In the 0th approximation, i.e. at $Mgl = 0$, the minimum of $U_{\text{ef}}$ is evidently achieved at $\cos \theta_0 = L_z/L_3$, turning the precession frequency (83) to zero. In the next, 1st approximation, we may require that at $\theta = \theta_0$, the derivative of the first term of Eq. (82) for $U_{\text{ef}}$ over $\cos \theta$ equal to -$L(A(L_z - L_3 \cos \theta)/I_A \sin^2 \theta)^{21}$ is canceled with that of the gravity-induced term, equal to $Mgl$. This immediately yields $\omega_{\text{pre}} = (L_z - L_3 \cos \theta_0)/I_A \sin^2 \theta_0 = Mgl/L_3$, so that identifying $\omega_{\text{rot}}$ with $\omega_3 \equiv L_3/I_3$ (see Fig. 8), we recover the simple expression (72).

The second important result that may be readily obtained from Eq. (82) is the exact expression for the threshold value of the spinning speed for a vertically rotating top ($\theta = 0$, $L_z = L_3$). Indeed, in the limit $\theta \to 0$ this expression may be readily simplified:

$$U_{\text{ef}}(\theta) \approx \text{const} + \left(\frac{L_3^2}{8I_A} - \frac{Mgl}{2}\right) \theta^2. \quad (4.84)$$

This formula shows that if $\omega_{\text{rot}} \equiv L_3/I_3$ is higher than the following threshold value,

\[\frac{L_3^2}{8I_A} - \frac{Mgl}{2} = \frac{L_3}{2Mgl},\]

21 Indeed, the derivative of the fraction $1/2I_A \sin^2 \theta$, taken at the point $\cos \theta = L_z/L_3$, is multiplied by the numerator, $(L_z - L_3 \cos \theta)^2$, which turns to zero at this point.
then the coefficient at $\theta^2$ in Eq. (84) is positive, so that $U_{ef}$ has a stable minimum at $\theta_0 = 0$. On the other hand, if $\omega_3$ is decreased below $\omega_{th}$, the fixed point becomes unstable, so that the top falls. As the plots in Fig. 11 show, Eq. (85) for the threshold frequency works very well even for non-zero but small values of the precession angle $\theta_0$. Note that if we take $I = I_A$ in the condition (73) of the approximate treatment, it acquires a very simple sense: $\omega_{rot} \gg \omega_{th}$.

Finally, Eqs. (82) give a natural description of one more phenomenon. If the initial energy is larger than $U_{ef}(\theta_0)$, the angle $\theta$ oscillates between two classical turning points on both sides of the fixed point $\theta_0$—see also Fig. 11. The law and frequency of these oscillations may be found exactly as in Sec. 3.3—see Eqs. (3.27) and (3.28). At $\omega_3 \gg \omega_{th}$, this motion is a fast rotation of the symmetry axis $n_3$ of the body about its average position performing the slow torque-induced precession. Historically, these oscillations are called *nutations*, but their physics is similar to that of the free precession that was analyzed in the previous section, and the order of magnitude of their frequency is given by Eq. (59).

It may be proved that small friction (not taken into account in the above analysis) leads first to decay of these nutations, then to a slower drift of the precession angle $\theta_0$ to zero and, finally, to a gradual decay of the spinning speed $\omega_{rot}$ until it reaches the threshold (85) and the top falls.

### 4.6. Non-inertial reference frames

To complete this chapter, let us use the results of our analysis of the rotation kinematics in Sec. 1 to complete the discussion of the transfer between two reference frames, started in the introductory Chapter 1. As Fig. 12 (which reproduces Fig. 1.2 in a more convenient notation) shows, even if the “moving” frame 0 rotates relative to the “lab” frame 0’, the radius vectors observed from these two frames are still related, at any moment of time, by the simple Eq. (1.5). In our new notation:

$$\mathbf{r}' = \mathbf{r}_0 + \mathbf{r}. \quad (4.86)$$

However, as was mentioned in Sec. 1, for velocities the general addition rule is already more complex. To find it, let us differentiate Eq. (86) over time:

$$\frac{d}{dt} \mathbf{r}' = \frac{d}{dt} \mathbf{r}_0 + \frac{d}{dt} \mathbf{r}. \quad (4.87)$$

The left-hand side of this relation is evidently the particle’s velocity as measured in the lab frame, and the first term on the right-hand side is the velocity $\mathbf{v}_0$ of the point 0, as measured in the same lab frame.
The last term is more complex: due to the possible mutual rotation of the frames 0 and 0’, that term may not vanish even if the particle does not move relative to the rotating frame 0 – see Fig. 12.

Fortunately, we have already derived the general Eq. (8) to analyze situations exactly like this one. Taking $\mathbf{A} = \mathbf{r}$ in it, we may apply the result to the last term of Eq. (87), to get

$$
\mathbf{v}_{\text{in lab}} = \mathbf{v}_0_{\text{in lab}} + (\mathbf{v} + \omega \times \mathbf{r}),
$$

(4.88)

where $\omega$ is the instantaneous angular velocity of an imaginary rigid body connected to the moving reference frame (or we may say, of this frame as such), as measured in the lab frame 0’, while $\mathbf{v}$ is $d\mathbf{r}/dt$ as measured in the moving frame 0. The relation (88), on one hand, is a natural generalization of Eq. (10) for $\mathbf{v} \neq 0$; on the other hand, if $\omega = 0$, it is reduced to simple Eq. (1.8) for the translational motion of the frame 0.

To calculate the particle’s acceleration, we may just repeat the same trick: differentiate Eq. (88) over time, and then use Eq. (8) again, now for the vector $\mathbf{A} = \mathbf{v} + \omega \times \mathbf{r}$. The result is

$$
\mathbf{a}_{\text{in lab}} \equiv \mathbf{a}_0_{\text{in lab}} + \frac{d}{dt}(\mathbf{v} + \omega \times \mathbf{r}) + \omega \times (\mathbf{v} + \omega \times \mathbf{r}).
$$

(4.89)

Carrying out the differentiation in the second term, we finally get the goal relation,

$$
\mathbf{a}_{\text{in lab}} \equiv \mathbf{a}_0_{\text{in lab}} + \mathbf{a} + \dot{\omega} \times \mathbf{r} + 2\omega \times \mathbf{v} + \omega \times (\omega \times \mathbf{r}),
$$

(4.90)

where $\mathbf{a}$ is particle’s acceleration, as measured in the moving frame. This result is a natural generalization of the simple Eq. (1.9) to the rotating frame case.

Now let the lab frame 0’ be inertial; then the 2nd Newton law for a particle of mass $m$ is

$$
ma_{\text{in lab}} = \mathbf{F},
$$

(4.91)

where $\mathbf{F}$ is the vector sum of all forces exerted on the particle. This is simple and clear; however, in many cases it is much more convenient to work in a non-inertial reference frame; for example, describing most phenomena on Earth’s surface, it is rather inconvenient to use a reference frame resting on the Sun (or in the galactic center, etc.). In order to understand what we should pay for the convenience of using a moving frame, we may combine Eqs. (90) and (91) to write

$$
ma = \mathbf{F} - ma_0_{\text{in lab}} - m\omega \times (\omega \times \mathbf{r}) - 2m\omega \times \mathbf{v} - m\omega \times \mathbf{r}.
$$

(4.92)

This result means that if we want to use a 2nd Newton law’s analog in a non-inertial reference frame, we have to add, to the actual net force $\mathbf{F}$ exerted on a particle, four pseudo-force terms, called inertial forces, all proportional to particle’s mass. Let us analyze them one by one, always remembering that these are just mathematical terms, not actual forces. (In particular, it would be futile to seek for the 3rd Newton law’s counterpart for any inertial force.)

The first term, $-ma_0_{\text{in lab}}$, is the only one not related to rotation and is well known from the undergraduate mechanics. (Let me hope the reader remembers all these weight-in-the-moving-elevator problems.) However, despite its simplicity, this term has more subtle consequences. As an example, let us consider, semi-qualitatively, the motion of a planet, such as our Earth, orbiting a star and also rotating about its own axis – see Fig. 13. The bulk-distributed gravity forces, acting on a planet from its star, are...
not quite uniform, because they obey the $1/r^2$ gravity law (1.15), and hence are equivalent to a single force applied to a point $A$ slightly offset from the planet’s center of mass $0$, toward the star. For a spherically-symmetric planet, the direction from $0$ to $A$ would be exactly aligned with the direction toward the star. However, real planets are not absolutely rigid, so that, due to the centrifugal “force” (to be discussed imminently), the rotation about their own axis makes them slightly ellipsoidal – see Fig. 13. (For our Earth, this *equatorial bulge* is about 10 km.) As a result, the net gravity force does create a small torque relative to the center of mass $0$. On the other hand, repeating all the arguments of this section for a body (rather than a point), we may see that, in the reference frame moving with the planet, the inertial force $-Ma_0$ (with the magnitude of the total gravity force, but directed *from* the star) is applied exactly to the center of mass and hence does not create a torque about it. As a result, this pair of forces creates a torque $\tau$ perpendicular to both the direction toward the star and the vector $0A$. (In Fig. 13, the torque vector is perpendicular to the plane of the drawing). If the angle $\delta$ between the planet’s “polar” axis of rotation and the direction towards the star was fixed, then, as we have seen in the previous section, this torque would induce a slow axis precession about that direction.

![Fig. 4.13. The axial precession of a planet (with the equatorial bulge and the 0A-offset strongly exaggerated).](image)

However, as a result of the orbital motion, the angle $\delta$ oscillates in time much faster (once a year) between values $(\pi/2 + \varepsilon)$ and $(\pi/2 - \varepsilon)$, where $\varepsilon$ is the axis tilt, i.e. angle between the polar axis (the direction of vectors $L$ and $\omega_{\text{rot}}$) and the normal to the *ecliptic plane* of the planet’s orbit. (For the Earth, $\varepsilon \approx 23.4^\circ$.) A straightforward averaging over these fast oscillations shows that the torque leads to the polar axis’ precession about the axis *perpendicular* to the ecliptic plane, keeping the angle $\varepsilon$ constant – see Fig. 13. For the Earth, the period $T_{\text{pre}} = 2\pi/\omega_{\text{pre}}$ of this *precession of the equinoxes*, corrected to a substantial effect of Moon’s gravity, is close to 26,000 years.

Returning to Eq. (92), the direction of the second term of its right-hand side,

$$\mathbf{F}_{\text{cf}} \equiv -m\omega \times (\omega \times \mathbf{r}),$$

(4.93)
called the *centrifugal force*, is always perpendicular to, and directed out of the instantaneous rotation axis – see Fig. 14. Indeed, the vector $\omega \times \mathbf{r}$ is perpendicular to both $\omega$ and $\mathbf{r}$ (in Fig. 14, normal to the drawing plane and directed from the reader) and has the magnitude $\omega \sin \theta = \rho \omega$, where $\rho$ is the distance of the particle from the rotation axis. Hence the outer vector product, with the account of the minus sign, is normal to the rotation axis $\omega$, directed from this axis, and is equal to $\omega^2 \sin \theta = \rho^2 \omega$. The “centrifugal

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22 Details of this calculation may be found, e.g., in Sec. 5.8 of the textbook by H. Goldstein *et al.*, *Classical Mechanics*, 3rd ed., Addison Wesley, 2002.

23 This effect is known from antiquity, apparently discovered by Hipparchus of Rhodes (190-120 BC).
force” is of course just the result of the fact that the centripetal acceleration \( \omega^2 \rho \), explicit in the inertial reference frame, disappears in the rotating frame. For a typical location of the Earth \((\rho \sim R_E \approx 6 \times 10^6 \text{ m})\), with its angular velocity \( \omega_E \approx 10^{-4} \text{ s}^{-1} \), the acceleration is rather considerable, of the order of 3 cm/s\(^2\), i.e. \( \sim 0.003 \text{ g} \), and is responsible, in particular, for the largest part of the equatorial bulge mentioned above.

As an example of using the centrifugal force concept, let us return again to our “testbed” problem on the bead sliding along a rotating ring – see Fig. 2.1. In the non-inertial reference frame attached to the ring, we have to add, to the actual forces \( mg \) and \( N \) acting on the bead, the horizontal centrifugal force\(^{24}\) directed from the rotation axis, with the magnitude \( m \omega^2 \rho \). Its component tangential to the ring equals \( (m \omega^2 \rho) \cos \theta = m \omega^2 R \sin \theta \cos \theta \), and hence the component of Eq. (92) along this direction is \( ma = -mg \sin \theta + m \omega^2 R \sin \theta \cos \theta \). With \( a = R \dot{\theta} \), this gives us an equation of motion equivalent to Eq. (2.25), which had been derived in Sec. 2.2 (in the inertial frame) using the Lagrangian formalism.

The third term on the right-hand side of Eq. (92),

\[
F_C = -2m \omega \times v ,
\]

is the so-called Coriolis force\(^{25}\), which is different from zero only if the particle moves in the rotating reference frame. Its physical sense may be understood by considering a projectile fired horizontally, say from the North Pole – see Fig. 15.

From the point of view of the Earth-based observer, the projectile will be affected by an additional Coriolis force (94), directed westward, with magnitude \( 2m \omega_E v \), where \( v \) is the main, southward component of the velocity. This force would cause the westward acceleration \( a = 2 \omega_E v \), and

\[24\] For this problem, all other inertial “forces”, besides the Coriolis force (see below) vanish, while the latter force is directed perpendicular to the ring and does not affect the bead’s motion along it.

\[25\] Named after G.-G. de Coriolis (already reverently mentioned in Chapter 1) who described its theory and applications in detail in 1835, though the first semi-quantitative analyses of this effect were given by Giovanni Battista Riccioli and Claude François Dechales already in the mid-1600s, and all basic components of the Coriolis theory may be traced to a 1749 work by Leonard Euler.
hence the westward deviation growing with time as \( d = \frac{a t^2}{2} = \omega_E v t^2 \). (This formula is exact only if \( d \) is much smaller than the distance \( r = vt \) passed by the projectile.) On the other hand, from the point of view of an inertial-frame observer, the projectile’s trajectory in the horizontal plane is a straight line. However, during the flight time \( t \), the Earth surface slips eastward from under the trajectory by the distance \( d = r \varphi = (vt)(\omega_E t) = \omega_E vt^2 \), where \( \varphi = \omega_E t \) is the azimuthal angle of the Earth rotation during the flight. Thus, both approaches give the same result – as they should.

Hence, the Coriolis “force” is just a fancy (but frequently very convenient) way of description of a purely geometric effect pertinent to the rotation, from the point of view of the observer participating in it. This force is responsible, in particular, for the higher right banks of rivers in the Northern hemisphere, regardless of the direction of their flow – see Fig. 16. Despite the smallness of the Coriolis force (for a typical velocity of the water in a river, \( v \sim 1 \text{ m/s} \), it is equivalent to acceleration \( a_C \sim 10^{-2} \text{ cm/s}^2 \sim 10^{-5} \text{ g} \)), its multi-century effects may be rather prominent.\(^{26}\)

Finally, the last, fourth term of Eq. (92), \(- m\dot{\omega} \times r\), exists only when the rotation frequency changes in time, and may be interpreted as a local-position-specific addition to the first term.

The key relation (92), derived above from the Newton equation (91), may be alternatively obtained from the Lagrangian approach, which gives, as a by-product, some important insights on the momentum, as well as on the relation between \( E \) and \( H \), at rotation. Let us use Eq. (88) to represent the kinetic energy of the particle in an inertial “lab” frame in terms of \( v \) and \( r \) measured in a rotating frame:

\[
T = \frac{m}{2} \left[ v_0 |_{\text{in\ lab}} + (v + \omega \times r) \right]^2 ,
\]

and use this expression to calculate the Lagrangian function. For the relatively simple case of a particle’s motion in the field of potential forces, measured from a reference frame that performs a pure rotation (so that \( v_0 |_{\text{in\ lab}} = 0 \))\(^{27}\) with a constant angular velocity \( \omega \), we get

\[
L \equiv T - U = \frac{m}{2} v^2 + m v \cdot (\omega \times r) + \frac{m}{2} (\omega \times r)^2 - U \equiv \frac{m}{2} v^2 + m v \cdot (\omega \times r) - U_{\text{ef}} ,
\]

\(^{26}\) The same force causes the counter-clockwise circulation in the “Nor’easter” storms on the US East Coast, having an the air velocity component directed toward the cyclone’s center, due to lower pressure in its middle.

\(^{27}\) A similar analysis of the cases with \( v_0 |_{\text{in\ lab}} \neq 0 \), for example, of a translational relative motion of the reference frames, is left for reader’s exercise.
where the effective potential energy,\(^\text{28}\)

\[ U_{\text{ef}} \equiv U + U_{\text{cf}}, \quad \text{with } U_{\text{ef}} \equiv -\frac{m}{2}(\omega \times r)^2, \]  

(4.96b)
is just the sum of the actual potential energy \(U\) of the particle and the so-called \textit{centrifugal potential energy}, associated with the centrifugal force (93):

\[ F_{\text{cf}} = -\nabla U_{\text{ef}} = -\nabla \left[ -\frac{m}{2}(\omega \times r)^2 \right] = -m\omega \times (\omega \times r). \]  

(4.97)

It is straightforward to verify that the Lagrangian equations of motion derived from Eqs. (96), considering the Cartesian components of \(r\) and \(v\) as generalized coordinates and velocities, coincide with Eq. (92) (with \(a_{\text{0}}|_{\text{lab}} = 0, \dot{\phi} = 0, \) and \(F = -\nabla U\)). Now it is very informative to have a look at a by-product of this calculation, the generalized momentum (2.31) corresponding to the particle’s coordinate \(r\) as measured in the rotating reference frame,\(^{29}\)

\[ \mathbf{p} \equiv \frac{\partial L}{\partial \dot{v}} = m(\mathbf{v} + \mathbf{\omega} \times \mathbf{r}). \]  

(4.98)

According to Eq. (88) with \(v_{\text{0}}|_{\text{lab}} = 0\), the expression in the parentheses is just \(v_{\text{in}}|_{\text{lab}}\). However, from the point of view of the moving frame, i.e. not knowing about the simple physical sense of the vector \(\mathbf{p}\), we would have a reason to speak about two different linear momenta of the same particle, the so-called \textit{kinetic momentum} \(p = mv\) and the \textit{canonical momentum} \(\mathbf{p} = p + m\mathbf{\omega} \times \mathbf{r}\).\(^{30}\)

Now let us calculate the Hamiltonian function \(H\), defined by Eq. (2.32), and the energy \(E\) as functions of the same moving-frame variables:

\[ H \equiv \sum_{j=1}^3 \frac{\partial L}{\partial \dot{v}_j} v_j - L = \mathbf{p} \cdot \mathbf{v} - L = m(\mathbf{v} + \mathbf{\omega} \times \mathbf{r}) \cdot \mathbf{v} - \left[ \frac{m}{2} v^2 + m\mathbf{v} \cdot (\mathbf{\omega} \times \mathbf{r}) - U_{\text{ef}} \right] = \frac{mv^2}{2} + U_{\text{ef}}, \]  

(4.99)

\[ E \equiv T + U = \frac{m}{2} v^2 + m\mathbf{v} \cdot (\mathbf{\omega} \times \mathbf{r}) + \frac{m}{2}(\mathbf{\omega} \times \mathbf{r})^2 + U = \frac{m}{2} v^2 + U_{\text{ef}} + m\mathbf{v} \cdot (\mathbf{\omega} \times \mathbf{r}) + m(\mathbf{\omega} \times \mathbf{r})^2. \]  

(4.100)

These expressions clearly show that \(E\) and \(H\) are \textit{not} equal.\(^{31}\) In hindsight, this is not surprising, because the kinetic energy (95), expressed in the moving-frame variables, includes a term linear in \(v\), and hence

\(^{28}\) For the attentive reader who has noticed the difference between the negative sign in the expression for \(U_{\text{ef}},\) and the positive sign before the similar second term in Eq. (3.44): as was already discussed in Chapter 3, this difference is due to the difference of assumptions. In the planetary problem, the angular momentum \(L\) (and hence its component \(L_z\)) is fixed, while the corresponding angular velocity \(\dot{\phi}\) is not. On the opposite, in our current discussion, the angular velocity \(\omega\) of the reference frame is assumed to be fixed, i.e. is independent of \(r\) and \(v\).

\(^{29}\) Here \(\partial L/\partial v\) is just a shorthand for a vector with Cartesian components \(\partial L/\partial v_j\). In a more formal language, this is the gradient of the scalar function \(L\) in the velocity space.

\(^{30}\) A very similar situation arises at the motion of a particle with electric charge \(q\) in magnetic field \(\mathcal{B}\). In that case, the role of the additional term \(\mathbf{p} = m\mathbf{\omega} \times \mathbf{r}\) is played by the product \(q\mathcal{A}\), where \(\mathcal{A}\) is the vector potential of the field \((\mathcal{B} = \nabla \times \mathcal{A})\) – see, e.g., EM Sec. 9.7, and in particular Eqs. (9.183) and (9.192).

\(^{31}\) Please note the last form of Eq. (99), which shows the physical sense of the Hamiltonian function of a particle in the rotating frame very clearly, as the sum of its kinetic energy (as measured in the moving frame), and the effective potential energy (96b), including that of the centrifugal “force”.

---

\[ U_{\text{ef}} \equiv U + U_{\text{cf}}, \quad \text{with } U_{\text{ef}} \equiv -\frac{m}{2}(\omega \times r)^2, \]  

(4.96b)
is not a quadratic-homogeneous function of this generalized velocity. The difference of these functions may be represented as

\[
E - H = m \mathbf{v} \cdot (\mathbf{\omega} \times \mathbf{r}) + m(\mathbf{\omega} \times \mathbf{r})^2 = m(\mathbf{v} + \mathbf{\omega} \times \mathbf{r}) \cdot (\mathbf{\omega} \times \mathbf{r}) = m\mathbf{v} \Big|_{\text{lab}} \cdot (\mathbf{\omega} \times \mathbf{r}).
\]

(4.101)

Now using the operand rotation rule again, we may transform this expression into a simpler form:

\[
E - H = \mathbf{\omega} \cdot (\mathbf{r} \times m\mathbf{v} \Big|_{\text{lab}}) = \mathbf{\omega} \cdot (\mathbf{r} \times \mathbf{\omega} \times \mathbf{r}) = \mathbf{\omega} \cdot \mathbf{L} \Big|_{\text{lab}}.
\]

(4.102)

As a sanity check, let us apply this general expression to the particular case of our testbed problem – see Fig. 2.1. In this case, the vector \( \mathbf{\omega} \) is aligned with the \( z \)-axis, so that of all Cartesian components of the vector \( \mathbf{L} \), only the component \( L_z \) is important for the scalar product in Eq. (102). This component evidently equals \( \omega I_z = \omega m R^2 = \omega m (R \sin \theta)^2 \), so that

\[
E - H = m \omega^2 R^2 \sin^2 \theta,
\]

(4.103)
i.e. the same result that follows from the subtraction of Eqs. (2.40) and (2.41).

### 4.7. Exercise problems

**4.1.** Calculate the principal moments of inertia for the following uniform rigid bodies:

(i) a thin, planar, round hoop, (ii) a flat round disk, (iii) a thin spherical shell, and (iv) a solid sphere. Compare the results, assuming that all the bodies have the same radius \( R \) and mass \( M \).

**4.2.** Calculate the principal moments of inertia for the rigid bodies shown in the figure below:

(i) an equilateral triangle made of thin rods with a uniform linear mass density \( \mu \), (ii) a thin plate in the shape of an equilateral triangle, with a uniform areal mass density \( \sigma \), and

---

32 Note that by the definition (1.36), the angular momenta \( \mathbf{L} \) of particles merely add up. As a result, the final form of Eq. (102) is valid for an arbitrary system of particles.
(iii) a tetrahedron made of a heavy material with a uniform bulk mass density $\rho$.

Assuming that the total mass of the three bodies is the same, compare the results and give an interpretation of their difference.

4.3. Prove that Eqs. (34)-(36) are valid for rotation of a rigid body about a fixed axis $z$, even if it does not pass through its center of mass.

4.4. The end of a uniform, thin rod of length $2l$ and mass $m$, initially at rest, is hit by a bullet of mass $m'$, flying with velocity $v_0$ (see the figure on the right), which gets stuck in the rod. Use two different approaches to calculate the velocity of the opposite end of the rod right after the collision.

4.5. A uniform ball is placed on a horizontal plane, while rotating with an angular velocity $\omega$, but having no initial linear velocity. Calculate the angular velocity after the ball’s slippage stops, assuming the Coulomb approximation for the kinetic friction force: $F_f = \mu N$, where $N$ is a pressure between the surfaces, and $\mu$ is a velocity-independent coefficient.

4.6. A body may rotate about a fixed horizontal axis $A$ – see Fig. 5. Find the frequency of its small oscillations, in a uniform gravity field, as a function of the distance $l$ of the axis from the body’s center of mass $0$, and analyze the result.

4.7. Calculate the frequency, and sketch the mode of oscillations of a round uniform cylinder of radius $R$ and the mass $M$, that may roll, without slipping, on a horizontal surface of a block of mass $M'$. The block, in turn, may move in the same direction, without friction, on a horizontal surface, being connected to it with an elastic spring – see the figure on the right.

4.8. A thin uniform bar of mass $M$ and length $l$ is hung on a light thread of length $l'$ (like a “chime” bell – see the figure on the right). Derive the equations of motion of the system within the plane of the drawing.

4.9. A solid, uniform, round cylinder of mass $M$ can roll, without slipping, over a concave, round cylindrical surface of a block of mass $M'$, in a uniform gravity field – see the figure on the right. The block can slide without friction on a horizontal surface. Using the Lagrangian formalism,

(i) find the frequency of small oscillations of the system near the equilibrium, and
(ii) sketch the oscillation mode for the particular case $M' = M$, $R' = 2R$.

4.10. A uniform solid hemisphere of radius $R$ is placed on a horizontal plane – see the figure on the right. Find the frequency of its small oscillations within a vertical plane, for two ultimate cases:
(i) there is no friction between the hemisphere and plane surfaces, and
(ii) the static friction is so strong that there is no slippage between these surfaces.

4.11. For the “sliding ladder” problem, started in Sec. 3 (see Fig. 7), find the critical value \( \alpha_c \) of the angle \( \alpha \) at that the ladder loses its contact with the vertical wall, assuming that it starts sliding from the vertical position, with a negligible initial velocity.

4.12. Six similar, uniform rods of length \( l \) and mass \( m \) are connected by light joints so that they may rotate, without friction, versus each other, forming a planar polygon. Initially, the polygon was at rest, and had the correct hexagon shape – see the figure on the right. Suddenly, an external force \( \mathbf{F} \) is applied to the middle of one rod, in the direction of the hexagon’s symmetry center. Calculate the accelerations: of the rod to which the force is applied (\( a \)), and of the opposite rod (\( a' \)), immediately after the application of the force.

4.13. A rectangular cuboid (parallelepiped) with sides \( a_1, a_2, \) and \( a_3 \), made of a material with a constant mass density \( \rho \), is rotated, with a constant angular velocity \( \omega \), about one of its space diagonals – see the figure on the right. Calculate the torque \( \tau \) necessary to sustain such rotation.

4.14. One end of a light shaft of length \( l \) is firmly attached to the center of a thin uniform solid disk of radius \( R << l \) and mass \( M \), whose plane is perpendicular to the shaft. Another end of the shaft is attached to a vertical axis (see the figure on the right) so that the shaft may rotate about the axis without friction. The disk rolls, without slippage, over a horizontal surface, so that the whole system rotates about the vertical axis with a constant angular velocity \( \omega \). Calculate the (vertical) supporting force \( N \) exerted on the disk by the surface.

4.15. An air-filled balloon is placed inside a water-filled container, which moves by inertia in free space, at negligible gravity. Suddenly, force \( \mathbf{F} \) is applied to the container, pointing in a certain direction. What direction does the balloon move relative to the container?

4.16. Two planets are in a circular orbit around their common center of mass. Calculate the effective potential energy of a much lighter body (say, a spacecraft) rotating with the same angular velocity, on the line connecting the planets. Sketch the plot of the radial dependence of \( U_{ef} \) and find out the number of so-called Lagrange points at which the potential energy has local maxima. Calculate their position explicitly in the limit when one of the planets is much more massive than the other one.

4.17. A small body is dropped down to the surface of Earth from height \( h << R_E \), without initial velocity. Calculate the magnitude and direction of its deviation from the vertical, due to the Earth rotation. Estimate the effect’s magnitude for a body dropped from the Empire State Building.
4.18. Calculate the height of solar tides on a large ocean, using the following simplifying assumptions: the tide period (½ of Earth's day) is much longer than the period of all ocean waves, the Earth (of mass $M_E$) is a sphere of radius $R_E$, and its distance $r_S$ from the Sun (of mass $M_S$) is constant and much larger than $R_E$.

4.19. A coin of radius $r$ is rolled, with velocity $V$, on a horizontal surface without slippage. What should be the coin's tilt angle $\theta$ (see the figure on the right) for it to roll on a circle of radius $R >> r$? Modeling the coin as a very thin, uniform disk, and assuming that the angle $\theta$ is small, solve this problem in:
(i) an inertial ("lab") reference frame, and
(ii) the non-inertial reference frame moving with the coin's center (but not rotating with it).

4.20. A satellite is on a circular orbit of radius $R$, around the Earth.
(i) Write the equations of motion of a small body as observed from the satellite, and simplify them for the case when the motion is limited to the satellite’s close vicinity.
(ii) Use the equations to prove that the body may be placed on an elliptical trajectory around the satellite’s center of mass, within its plane of rotation about Earth. Calculate the ellipse’s orientation and eccentricity.

4.21. A non-spherical shape of an artificial satellite may ensure its stable angular orientation relative to Earth’s surface, advantageous for many practical goals. Modeling the satellite as a strongly elongated, axially-symmetric body, moving around the Earth on a circular orbit of radius $R$, find its stable orientation.

4.22.* A rigid, straight, uniform rod of length $l$, with the lower end on a pivot, falls in a uniform gravity field – see the figure on the right. Neglecting friction, calculate the distribution of the bending torque $\tau$ along its length, and analyze the result.

*Hint:* The bending torque is the net torque of the force $F$ acting between two parts of the rod, mentally separated by its cross-section, about a certain “neutral axis”. As will be discussed in detail in Sec. 7.5, at the proper definition of this axis, the bending torque’s gradient along the rod’s length is equal to $(-F)$, where $F$ is the rod-normal (“shear”) component of the force exerted by the top part of the rod on its lower part.

4.23. Let $\mathbf{r}$ be the radius vector of a particle, as measured in a possibly non-inertial but certainly non-rotating reference frame. Taking its Cartesian components for the generalized coordinates, calculate the corresponding generalized momentum $\mathbf{p}$ of the particle, and its Hamiltonian function $H$. Compare $\mathbf{p}$ with $m\mathbf{v}$, and $H$ with the particle’s energy $E$. Derive the Lagrangian equation of motion in this approach, and compare it with Eq. (92).

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33 Inadequate definitions of this torque are the main reason for numerous wrong solutions of this problem, posted online – readers beware!