Chapter 5. Magnetism

Despite the fact that we are now starting to discuss a completely new type of electromagnetic interactions, its coverage (for the stationary case) will take just one chapter, because we will be able to recycle many ideas and methods of electrostatics, though with a twist or two.

5.1. Magnetic interaction of currents

DC currents in conductors usually leave them *electroneutral*, $\rho(\mathbf{r}) = 0$, with a very good precision, because any virtual misbalance of positive and negative charge density results in extremely strong Coulomb forces that restore their balance by an additional shift of free carriers.[1] This is why let us start the discussion of magnetic interactions from the simplest case of two spatially-separated, current-carrying, electroneutral conductors (Fig. 1).

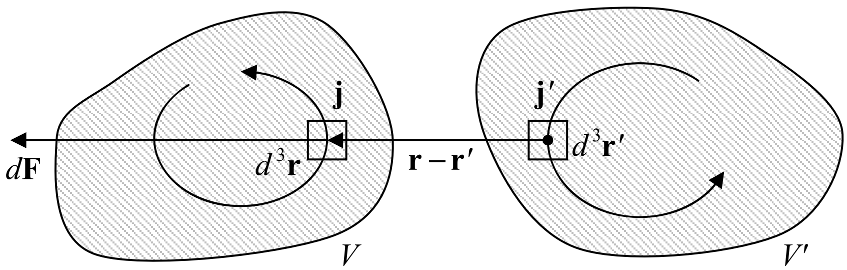

Fig. 5.1. Magnetic interaction of two currents.

According to the Coulomb law, there should be no force between them. However, several experiments carried out in the early 1820s[2] proved that such non-Coulomb forces do exist, and are the manifestation of another, *magnetic* interactions between the currents. In the contemporary used in this course, their results may be summarized with one formula, in SI units expressed as:[3]

Magnetic force between currents

$$\mathbf{F} = -\frac{\mu_0}{4\pi}\int_V d^3r \int_{V'} d^3r' \left(\mathbf{j}(\mathbf{r})\cdot\mathbf{j}'(\mathbf{r}')\right)\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}. \tag{5.1}$$

Here coefficient $\mu_0/4\pi$ (where μ_0 is called either the *magnetic constant* or the *free space permeability*), by definition, equals *exactly* 10^{-7} SI units, thus relating the electric current (and hence electric charge) definition to that of force – see below.

Note that the Coulomb law (1.1), with the account of the linear superposition principle, may be presented in a very similar form:

[1] The most important case when the electroneutrality does not hold is the motion of electrons in vacuum. In this case, magnetic forces coexist with (typically, stronger) electrostatic forces – see Eq. (3) below and its discussion. In some semiconductor devices, local violations of electroneutrality also play an important role.

[2] Most notably, by H. C. Ørsted, J.-B. Biot and F. Savart, and A.-M. Ampère.

[3] In the Gaussian units, coefficient $\mu_0/4\pi$ is replaced with $1/c^2$ (i.e., implicitly with $\mu_0\varepsilon_0$) where c is the speed of light, in modern metrology considered *exactly* known – see, e.g., appendix *CA: Selected Physical Constants*.

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\int_V d^3r \int_{V'} d^3r' \; \rho(\mathbf{r})\rho'(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}. \qquad (5.2)$$

Besides the different coefficient and sign, the "only" difference of Eq. (1) from Eq. (2) is the scalar product of current densities, evidently necessary because of the vector character of the current density. We will see that this difference will bring certain complications in applying the electrostatics approaches, discussed in the previous chapters, to magnetostatics.

Before going to their discussion, let us have one more glance at the coefficients in Eqs. (1) and (2). To compare them, let us consider two objects with uncompensated charge distributions $\rho(\mathbf{r})$ and $\rho'(\mathbf{r})$, each moving parallel to each other as a whole certain velocities $\mathbf{v}$ and $\mathbf{v}'$, as measured in an inertial "lab" frame. In this case, $\mathbf{j}(\mathbf{r}) = \rho(\mathbf{r})\mathbf{v}$, $\mathbf{j}(\mathbf{r})\cdot\mathbf{j}'(\mathbf{r}) = \rho(\mathbf{r})\rho'(\mathbf{r})vv'$, and the integrals in Eqs. (1) and (2) become functionally similar, and differ only by the factor

$$\frac{F_{\text{magnetic}}}{F_{\text{electric}}} = -\frac{\mu_0 vv'}{4\pi} \Big/ \frac{1}{4\pi\varepsilon_0} = -\frac{vv'}{c^2}. \qquad (5.3)$$

(This expression hold in any consistent system of units.) We immediately see that magnetism is an essentially relativistic phenomenon, very weak in comparison with the electrostatic interaction at the human scale velocities, $v \ll c$, and may dominate only if the latter interaction vanishes – as it does in electroneutral systems.[4]

Also, Eq. (3) points at an interesting paradox. Consider two electron beams moving parallel to each other, with the same velocity v with respect to a lab reference frame. Then, according to Eq. (3), the net force of their total (electric and magnetic) interaction is proportional to $(1 - v^2/c^2)$, and tends to zero in the limit $v \to c$. However, in the reference frame moving together with electrons, they are not moving at all, i.e. $v = 0$. Hence, from the point of view of such a moving observer, the electron beams should interact only electrostatically, with a repulsive force independent of velocity v. Historically, this had been one of several paradoxes that led to the development of the special relativity; its resolution will be discussed in Chapter 9, devoted to this theory.

Returning to Eq. (1), in some simple cases, the double integration in it may be carried out analytically. First of all, let us simplify this expression for the case of two thin, long conductors (wires) separated by a distance much larger than their thickness. In this case we may integrate the products $\mathbf{j}d^3r$ and $\mathbf{j}'d^3r'$ over wires' cross-sections first, neglecting the corresponding change of $(\mathbf{r} - \mathbf{r}')$. Since the integrals of the current density over the cross-sections of the wire are just the currents I and I' in the wires, and cannot change along their lengths (correspondingly, l and l'), they may be taken out of the remaining integrals, reducing Eq. (1) to

$$\mathbf{F} = -\frac{\mu_0 II'}{4\pi}\oint_l \oint_{l'} \left(d\mathbf{r}\cdot d\mathbf{r}'\right)\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}. \qquad (5.4)$$

[4] The discovery and initial studies of such a subtle, relativistic phenomenon as magnetism in the early 19th century was much facilitated by the relative abundance of natural *ferromagnets*, materials with spontaneous magnetic polarization, whose strong magnetic field may be traced back to relativistic effects (such as spin) in atoms. (The electrostatic analogs of such materials, *electrets*, are much more rare.) I will briefly discuss the ferromagnetism in Sec. 5 below.

As the simplest example, consider two straight, parallel wires (Fig. 2), separated by distance d, with length $l >> \rho$. In this case, due to symmetry, the vector of magnetic interaction force has to:

(i) lay in the same plane as the currents, and
(ii) be perpendicular to the wires – see Fig. 2.

Hence we can limit our calculations to just one component of the force. Using the fact that with the coordinate choice shown in Fig. 2, $d\mathbf{r}\cdot d\mathbf{r}' = dxdx'$, we get

$$F = -\frac{\mu_0 II'}{4\pi}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' \frac{\sin\theta}{d^2 + (x-x')^2} = -\frac{\mu_0 II'}{4\pi}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dx' \frac{d}{\left[d^2 + (x-x')^2\right]^{3/2}}. \qquad (5.5)$$

Introducing, instead of x', a new, dimensionless variable $\xi \equiv (x - x')/\rho$, we may reduce the internal integral to a table integral which we have already met in this course:

$$F = -\frac{\mu_0 II'}{4\pi d}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} \frac{d\xi}{\left(1+\xi^2\right)^{3/2}} = -\frac{\mu_0 II'}{2\pi d}\int_{-\infty}^{+\infty} dx. \qquad (5.6)$$

The integral over x is formally diverging, but this means merely that the interaction force *per unit length* of the wires is constant:

$$\frac{F}{l} = -\frac{\mu_0 II'}{2\pi d} \quad . \qquad (5.7)$$

Note that the force drops rather slowly (only as $1/d$) as the distance d between the wires is increased, and is *attractive* (rather than repulsive as in the Coulomb law) if the currents are of the same sign.

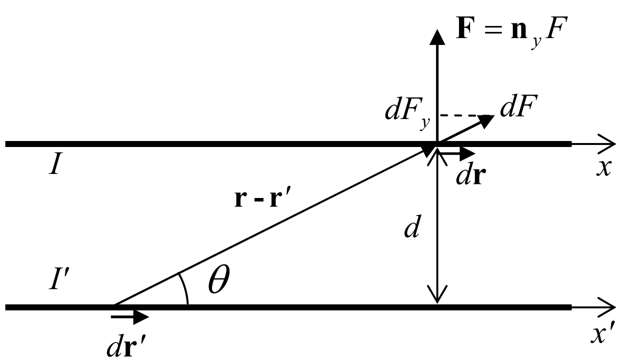

Fig. 5.2. Magnetic force between two straight parallel currents.

This is an important result,[5] but again, the problems solvable so simply are few and far between, and it is intuitively clear that we would strongly benefit from the same approach as in electrostatics, i.e., from breaking Eq. (1) into a product of two factors via the introduction of a suitable *field*. Such decomposition may done as follows:

Lorentz force on a current

$$\mathbf{F} = \int_V \mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r}) d^3r, \qquad (5.8)$$

[5] In particular, Eq. (7) is used for the legal definition of the SI unit of current, one *ampere* (A), via the SI unit of force (the newton, N), with coefficient μ_0 fixed as listed above.

where vector **B** is called the *magnetic field* (in our particular case, induced by current **j** ’):[6]

Biot-Savart law

$$\mathbf{B}(\mathbf{r}) \equiv \frac{\mu_0}{4\pi}\int_{V'} \mathbf{j}'(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} d^3r' . \qquad (5.9)$$

The last equation is called the *Biot-Savart law*, while **F** expressed by Eq. (8) is sometimes called the *Lorentz force*.[7] However, more frequently the later term is reserved for the full force,

Lorentz force on a particle

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right), \qquad (5.10)$$

exerted by electric and magnetic fields field on a point charge q, moving with velocity **v**. (The equivalence of Eq. (8) and the magnetic part of Eq. (10) follows from the summation of all forces acting on n particles in a unit volume, moving with the same velocity **v**, so that $\mathbf{j} = qn\mathbf{v}$.)

Now we have to prove that the new formulation (8)-(9) is equivalent to Eq. (1). At the first glance, this seems unlikely. Indeed, first of all, Eqs. (8) and (9) involve vector products, while Eq. (1) is based on a scalar product. More profoundly, in contrast to Eq. (1), Eqs. (8) and (9) do *not* satisfy the 3rd Newton’s law, applied to elementary current components $\mathbf{j}d^3r$ and $\mathbf{j}'d^3r'$, if these vectors are not parallel to each other. Indeed, consider the situation shown in Fig. 3. Here vector **j** ’ is perpendicular to vector ($\mathbf{r} - \mathbf{r}'$), and hence, according to Eq. (9), produces a nonvanishing contribution $d\mathbf{B}'$ to the magnetic field, directed (in Fig. 3) perpendicular to the plane of drawing, i.e. is perpendicular to vector **j**. Hence, according to Eq. (8), this field provides a nonvanishing contribution to **F**. On the other hand, if we calculate the reciprocal force **F** ’ by swapping indices in Eqs. (8) and (9), the latter equation immediately shows that $d\mathbf{B}(\mathbf{r}') \propto \mathbf{j}\times(\mathbf{r} - \mathbf{r}') = 0$, because the two operand vectors are parallel (Fig. 3). Hence, the current component $\mathbf{j}'d^3\mathbf{r}'$ does exert a force on its counterpart, while $\mathbf{j}d^3\mathbf{r}$ does not.

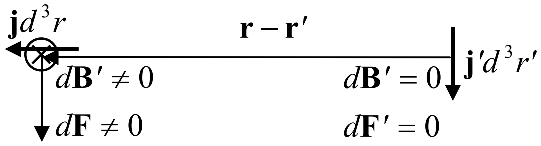

Fig. 5.3. Apparent violation of the 3rd Newton law in magnetism.

Despite this apparent problem, let us still go ahead and plug Eq. (9) into Eq. (8):

$$\mathbf{F} = \frac{\mu_0}{4\pi}\int_V d^3r \int_{V'} d^3r' \, \mathbf{j}(\mathbf{r}) \times \left(\mathbf{j}'(\mathbf{r}') \times \frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3} \right). \qquad (5.11)$$

[6] The SI unit of the magnetic field is called *tesla*, T - after N. Tesla, an electrical engineering pioneer. In the Gaussian units, the already discussed constant $1/c^2$ in Eq. (1) is equally divided between Eqs. (8) and (9), so that in them both, the constant before the integral is $1/c$. The resulting Gaussian unit of field **B** is called *gauss* (G); taking into account the difference of units of electric charge and length, and hence current density, 1 G equals exactly 10^{-4} T. Note also that in some textbooks, especially old ones, **B** is called either the *magnetic induction*, or the *magnetic flux density*, while the term “magnetic field” is reserved for vector **H** that will be introduced Sec. 5 below.

[7] Named after H. Lorentz, who received a Nobel prize for his explanation of the Zeeman effect, but is more famous for his numerous contributions to the development of special relativity – see Chapter 9. To be fair, the magnetic part of the Lorentz force was correctly calculated first by O. Heaviside.

This double vector product may transformed into two scalar products, using the vector algebraic identity called the *bac minus cab rule*, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{a}\cdot\mathbf{c}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b})$.[8] Applying this relation, with $\mathbf{a} = \mathbf{j}$, $\mathbf{b} = \mathbf{j}'$, and $\mathbf{c} = \mathbf{R} \equiv \mathbf{r} - \mathbf{r}'$, to Eq. (11), we get

$$\mathbf{F} = \frac{\mu_0}{4\pi}\int_{V'} d^3r'\mathbf{j}'(\mathbf{r}')\left(\int_V d^3r \frac{\mathbf{j}(\mathbf{r})\cdot\mathbf{R}}{R^3}\right) - \frac{\mu_0}{4\pi}\int_V d^3r\int_{V'} d^3r'\, \mathbf{j}(\mathbf{r})\cdot\mathbf{j}'(\mathbf{r}')\frac{\mathbf{R}}{R^3}. \qquad (5.12)$$

The second term in the right-hand part of this equation coincides with the right-hand part of Eq. (1), while the first term equals zero, because its the internal integral vanishes. Indeed, we may break volumes V and V' into narrow *current tubes*, the stretched sub-volumes whose walls are not crossed by current lines ($j_n = 0$). As a result, the (infinitesimal) current in each tube, $dI = jdA = jd^2r$, is the same along its length, and, just as in a thin wire, $\mathbf{j}d^2r$ may be replaced with $dId\mathbf{r}$. Because of this, each tube's contribution to the internal integral in the first term of Eq. (12) may be presented as

$$dI\oint_l d\mathbf{r}\cdot\frac{\mathbf{R}}{R^3} = -dI\oint_l d\mathbf{r}\cdot\nabla\frac{1}{R} = -dI\oint_l dr\frac{\partial}{\partial r}\frac{1}{R}, \qquad (5.13)$$

where operator ∇ acts in the $\mathbf{r}$ space, and the integral is taken along tube's length l. Due to the current continuity, each loop should follow a closed contour, and an integral of a full differential of some scalar function (in our case, $1/r_{12}$) along it equals zero.

So we have recovered Eq. (1). Returning for a minute to the paradox illustrated with Fig. 3, we may conclude that the apparent violation of the 3rd Newton law was the artifact of our interpretation of Eqs. (8) and (9) as sums of independent elementary components. In reality, due to the dc current continuity expressed by Eq. (4.6), these components are *not* independent. For the whole currents, Eqs. (8)-(9) do obey the 3rd law – as follows from their already proved equivalence to Eq. (1).

Thus we have been able to break the magnetic interaction into the two effects: the creation of the *magnetic field* $\mathbf{B}$ by one current (in our notation, $\mathbf{j}'$), and the effect of this field on the other current ($\mathbf{j}$). Now comes an additional experimental fact: other elementary components $\mathbf{j}d^3r'$ of current $\mathbf{j}$ also contribute to the magnetic field (9) acting on component $\mathbf{j}d^3r$.[9] This fact allows us to drop prime after $\mathbf{j}$ in Eq. (9), and rewrite Eqs. (8) and (9) as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{V'} \mathbf{j}(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{\left|\mathbf{r}-\mathbf{r}'\right|^3}d^3r', \qquad (5.14)$$

$$\mathbf{F} = \int_V \mathbf{j}(\mathbf{r})\times\mathbf{B}(\mathbf{r})d^3r, \qquad (5.15)$$

Again, the field *observation* point $\mathbf{r}$ and the field *source* point $\mathbf{r}'$ have to be clearly distinguished. We immediately see that these expressions are similar to, but still different from the corresponding relations of the electrostatics, namely Eq. (1.8),

[8] See, e.g., MA Eq. (7.5).

[9] Just in electrostatics, one needs to exercise due caution at transfer from these expressions to the limit of discrete classical particles, and extended wavefunctions in quantum mechanics, in order to avoid the (non-existing) magnetic interaction of a charged particle upon itself.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\oint_{V'} \rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} d^3r' , \tag{5.16}$$

and the distributed version of Eq. (1.6):

$$\mathbf{F} = \oint_V \rho(\mathbf{r})\,\mathbf{E}(\mathbf{r}) d^3r \ . \tag{5.17}$$

(Note that the sign difference has disappeared, at the cost of the replacement of scalar-by-vector multiplications in electrostatics with cross-products of vectors in magnetostatics.)

For the frequent case of a field of a thin wire of length *l'*, Eq. (14) may be re-written as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint_{l'} d\mathbf{r}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} . \tag{5.18}$$

Let us see how does the last formula work for the simplest case of a straight wire (Fig. 4a). The magnetic field contribution $d\mathbf{B}$ due to any small fragment $d\mathbf{r}'$ of the wire's length is directed along the same line (perpendicular to both the wire and the perpendicular d dropped from the observation point to the wire line), and its magnitude is

$$dB = \frac{\mu_0 I}{4\pi}\frac{dx'}{|\mathbf{r}-\mathbf{r}'|^2}\sin\theta = \frac{\mu_0 I}{4\pi}\frac{dx'}{\left(d^2+x^2\right)}\frac{d}{\left(d^2+x^2\right)^{1/2}} . \tag{5.19}$$

Summing up all such contributions, we get

$$B = \frac{\mu_0 I \rho}{4\pi}\int_{-\infty}^{\infty}\frac{dx}{(x^2+d^2)^{3/2}} = \frac{\mu_0 I}{2\pi d} . \tag{5.20}$$

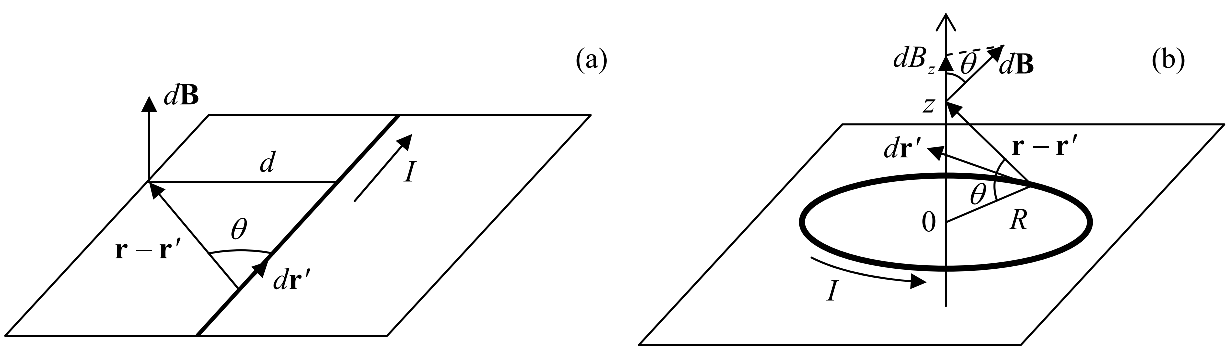

Fig. 5.4. Magnetic fields of: (a) a straight current, and (b) a current loop.

This is a simple but very important result. (Note that it is only valid for very long ($l >> d$), straight wires.) It is especially crucial to note the "vortex" character of the field: its lines go around the wire, forming round rings with the centers on the current line. This is in the sharp contrast to the electrostatic field lines that can only begin and end on electric charges and never form closed loops (otherwise the Coulomb force $q\mathbf{E}$ would not be conservative). In the magnetic case, the vortex *field* may be reconciled with the potential character of magnetic *forces*, which is evident from Eq. (1), due to the vector products in Eqs. (14)-(15).

Now we may use Eq. (15), or rather its thin-wire version

$$\mathbf{F} = I\oint_l d\mathbf{r} \times \mathbf{B}(\mathbf{r}), \tag{5.21}$$

to apply Eq. (20) to the two-wire problem (Fig. 2). Since for the second wire vectors d**r** and **B** are perpendicular to each other, we immediately arrive at our previous result (7).

The next important application of the Biot-Savart law (14) is the magnetic field at the axis of a circular current loop (Fig. 4b). Due to the problem symmetry, the net field **B** has to be directed along the axis, but each of its components d**B** is tilted by angle $\theta = \tan^{-1}(z/R)$ to this axis, so that its axial component

$$dB_z = dB\cos\theta = \frac{\mu_0 I}{4\pi}\frac{dr'}{R^2+z^2}\frac{R}{\left(R^2+z^2\right)^{1/2}}. \tag{5.22}$$

Since the denominator of this expression remains the same for all wire components dr', in this case the integration is trivial ($\int dr' = 2\pi R$), giving finally

$$B = \frac{\mu_0 I}{2}\frac{R^2}{\left(R^2+z^2\right)^{3/2}}. \tag{5.23}$$

Note that the magnetic field in the loop's center (i.e., for $z = 0$),

$$B = \frac{\mu_0 I}{2R}, \tag{5.24}$$

is π times higher than that due to a similar current in a straight wire, at distance $d = R$ from it. This increase it readily understandable, since all elementary components of the loop are at the same distance R from the observation point, while in the case of a straight wire, all its point but one are separated from the observation point by a distance larger than d.

Another notable fact is that at large distances ($z^2 >> R^2$), field (23) is proportional to z^{-3}:

$$B \approx \frac{\mu_0 I}{2}\frac{R^2}{|z|^3} = \frac{\mu_0}{4\pi}\frac{2m}{|z|^3}, \tag{5.25}$$

just like the electric field of a dipole (along its direction), with the replacement of the electric dipole moment magnitude p with $m = IA$, where $A = \pi R^2$ is the loop area. This is the best example of a *magnetic dipole*, with *dipole moment m* - the notions to be discussed in more detail in Sec. 5 below.

5.2. Vector-potential and the Ampère law

The reader can see that the calculations of the magnetic field using Eq. (14) or (18) are still cumbersome even for the very simple systems we have examined. As we saw in Chapter 1, similar calculations in electrostatics, at least for several important systems of high symmetry, could be substantially simplified using the Gauss law (1.16). A similar relation exists in magnetostatics as well, but has a different form, due to the vortex character of the magnetic field. To derive it, let us notice that in an analogy with the scalar case, the vector product under integral (14) may be transformed as

$$\frac{j(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}=\nabla\times\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{5.26}$$

where operator ∇ acts in the $\mathbf{r}$ space. (This equality may be really verified by its Cartesian components, noticing that the current density is a function of $\mathbf{r}'$ and hence its components are independent of $\mathbf{r}$.) Plugging Eq. (26) into Eq. (14), and moving operator ∇ out of the integral over $\mathbf{r}'$, we see that the magnetic field may be presented as the curl of another vector field:[10]

$$\mathbf{B}(\mathbf{r})=\nabla\times\mathbf{A}(\mathbf{r}), \tag{5.27}$$

namely the so-called *vector-potential*:

Vector-potential

$$\mathbf{A}(\mathbf{r})\equiv\frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'. \tag{5.28}$$

Please note a wonderful analogy between Eqs. (27)-(28) and, respectively, Eqs. (1.33) and (1.38). This analogy implies that vector-potential $\mathbf{A}$ plays, for the magnetic field, essentially the same role as the scalar potential ϕ plays for the electric field (hence the name "potential"), with due respect to the vortex character of $\mathbf{A}$. I will discuss this notion in detail below.

Now let us see what equations we may get for the spatial derivatives of the magnetic field. First, vector algebra says that the divergence of any curl is zero.[11] In application to Eq. (27), this means that

No magnetic monopoles

$$\nabla\cdot\mathbf{B}=0. \tag{5.29}$$

Comparing this equation with Eq. (1.27), we see that Eq. (29) may be interpreted as the absence of a magnetic analog of an electric charge on which magnetic field lines could originate or end. Numerous searches for such hypothetical magnetic charges, called *magnetic monopoles*, using very sensitive and sophisticated experimental setups, have never given a convincing evidence of their existence in Nature.

Proceeding to the alternative, vector derivative of the magnetic field (i.e., its curl), and using Eq. (28), we get

$$\nabla\times\mathbf{B}(\mathbf{r})=\frac{\mu_0}{4\pi}\nabla\times\left(\nabla\times\int_{V'}\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'\right). \tag{5.30}$$

This expression may be simplified by using the following general vector identity:[12]

$$\nabla\times(\nabla\times\mathbf{c})=\nabla(\nabla\cdot\mathbf{c})-\nabla^2\mathbf{c}, \tag{5.31}$$

applied to vector $\mathbf{c}(\mathbf{r})=\mathbf{j}(\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|$:

$$\nabla\times\mathbf{B}=\frac{\mu_0}{4\pi}\nabla\int_{V'}\mathbf{j}(\mathbf{r}')\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r'-\frac{\mu_0}{4\pi}\int_{V'}\mathbf{j}(\mathbf{r}')\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^3r'. \tag{5.32}$$

As was already discussed during our study of electrostatics,

[10] In the Gaussian units, Eq. (27) remains the same, and hence in Eq. (28), coefficient $\mu_0/4\pi$ is replaced with $1/c$.
[11] See, e.g., MA Eq. (11.2).
[12] See, e.g., MA Eq. (11.3).

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}'), \tag{5.33}$$

so that the last term of Eq. (32) is just $\mu_0\mathbf{j}(\mathbf{r})$. On the other hand, inside the first integral we can replace ∇ with $(-\nabla')$, where prime means differentiation in the space of radius-vector $\mathbf{r}'$. Integrating that term by parts, we get

$$\nabla\times\mathbf{B} = -\frac{\mu_0}{4\pi}\nabla\oint_{S'} j_n(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|}d^2r' + \nabla\int_{V'}\frac{\nabla'\cdot\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r' + \mu_0\mathbf{j}(\mathbf{r}). \tag{5.34}$$

Applying this equation to the volume V' limited by a surface S' sufficiently distant from the field concentration (or with no current crossing it), we may neglect the first term in the right-hand part of Eq. (34), while the second term always equals zero in statics, due to the dc charge continuity – see Eq. (4.6). As a result, we arrive at a very simple differential equation[13]

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j}. \tag{5.35}$$

This is (the dc form of) the inhomogeneous Maxwell equation, which in magnetostatics plays the role similar to the Poisson equation (1.27) in electrostatics. Let me display, for the first time in this course, this fundamental system of equations (at this stage, for statics only), and give the reader a minute to stare at their beautiful symmetry - that has inspired so much of the 20th century physics:

Static Maxwell equations

$$\nabla\times\mathbf{E} = 0, \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{j},$$
$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}, \qquad \nabla\cdot\mathbf{B} = 0. \tag{5.36}$$

Their only asymmetry, two zeros in the right hand parts (for the magnetic field's divergence and electric field's curl), is due to the absence in Nature of, respectively, the magnetic monopoles and their currents. I will discuss these equations in more detail in Sec. 6.7, after the equations for field curls have been generalized to their full (time-dependent) versions.

Returning now to a more mundane but important task of calculating magnetic field induced by simple current configurations, we can benefit from an integral form of Eq. (35). For that, let us integrate this equation over an arbitrary surface S limited by a closed contour C, applying to it the Stokes theorem.[14] The resulting expression,

Ampère law

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = \mu_0\oint_S j_n d^2r \equiv \mu_0 I, \tag{5.37}$$

where I is the net electric current crossing surface S, is called the *Ampère law*.

As the first example of its application, let us return to a current in a straight wire (Fig. 4a). With the Ampère law in our arsenal, we can readily pursue an even more ambitious goal – calculate the magnetic field both outside and inside of a wire of arbitrary radius R, with an arbitrary (albeit axially-symmetric) current distribution $j(\rho)$ – see Fig. 5. Selecting two contours C in the form of rings of some

[13] As in all earlier formulas for the magnetic field, in the Gaussian units the coefficient μ_0 in this relation has to be replaced with $4\pi/c$.

[14] See, e.g., MA Eq. (12.1) with $\mathbf{f} = \mathbf{B}$.

radius ρ in the plane perpendicular to the wire axis z, we have $\mathbf{B}\cdot d\mathbf{r} = B\rho(d\varphi)$, these φ is the azimuthal angle, so that the Ampère law (37) yields:

$$2\pi\,\rho B = \mu_0 \times \begin{cases} 2\pi \int_0^{\rho} j(\rho')\rho' d\rho', & \text{for } \rho \le R, \\ 2\pi \int_0^{R} j(\rho')\rho' d\rho' \equiv I, & \text{for } \rho \ge R. \end{cases} \qquad (5.38)$$

Thus we have not only recovered our previous result (20), with the notation replacement $d \to \rho$, in a much simpler way, but could also find the magnetic field distribution inside the wire. (In the most common case when the wire conductivity σ is constant, and hence the current is uniformly distributed along its cross-section, $j(\rho)$ = const, the first of Eqs. (38) immediately yields $B \propto \rho$ for $\rho \le R$).

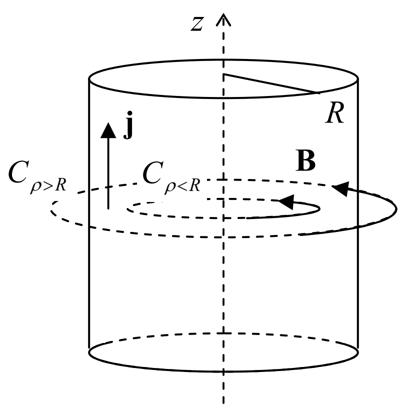

Fig. 5.5. The simplest application of the Ampère law: dc current in a straight wire.

Another important example is a straight, long *solenoid* (Fig. 6a), with dense winding: $n^2A >> 1$, where n is the number of wire turns per unit length and A is the area of solenoid's cross-section - not necessarily circular.

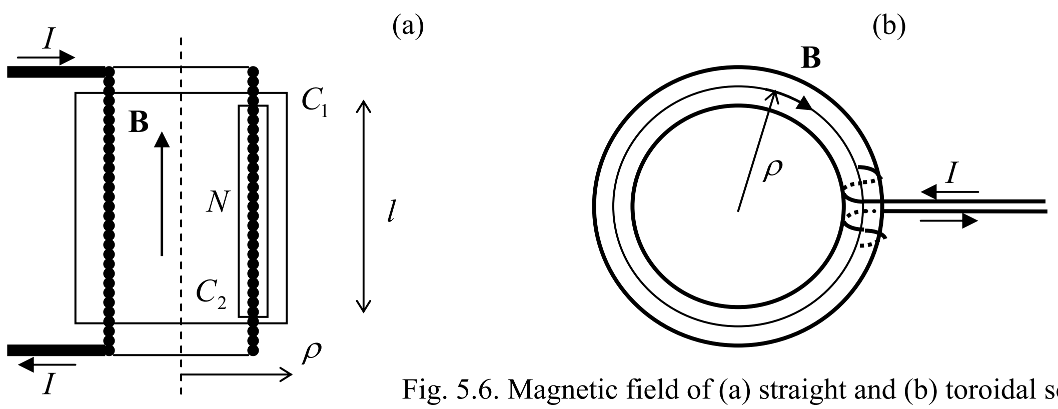

Fig. 5.6. Magnetic field of (a) straight and (b) toroidal solenoids.

From the symmetry of this problem, the longitudinal (in Fig. 6a, vertical) component B_z of the magnetic field may only depend on the horizontal position ρ of the observation point. First taking a plane Ampère contour C_1, with both long sides outside the solenoid, we get $B_z(\rho_2) - B_z(\rho_1) = 0$, because the total current piercing the contour equals zero. This is only possible if $B_z = 0$ at any ρ outside of the (infinitely long!) solenoid.[15] With this result on hand, from contour C_2 we get the following relation for the only (z-) component of the internal field:

$$Bl = \mu_0 NI \,, \tag{5.39}$$

where N is the number of wire turns passing through the contour of length l. This means that regardless of the exact position internal side of the contour, the result is the same:

$$B = \mu_0 \frac{N}{l} I = \mu_0 nI \,. \tag{5.40}$$

Thus, the field inside an infinitely long solenoid is uniform; in this sense, a long solenoid is a magnetic analog of a wide plane capacitor.

As should be clear from its derivation, the obtained result, especially that the field outside of the solenoid equals zero, is conditional on the solenoid length being very large in comparison with its lateral size. (From Eq. (25), we may predict that for a solenoid of a finite length l, the external field is only a factor of $\sim A/l^2$ lower than the internal one.) Much better suppression of this external ("fringe") field may be obtained using the *toroidal solenoid* (Fig. 6b). The application of Ampère law to this geometry shows that, in the limit of dense winding ($N >> 1$), there is no fringe field at all – for any relation between two radii of the thorus, while inside the solenoid, and distance ρ from the center,

$$B = \frac{\mu_0 NI}{2\pi\rho} \,. \tag{5.41}$$

We see that a possible drawback of this system for practical applications is that internal field depends on ρ, i.e. is not quite uniform; however, if the thorus is thin, this problem is minor.

How should we solve the problems of magnetostatics for systems whose low symmetry does not allow getting easy results from the Ampère law? (The examples are of course too numerous to list; for example, we cannot use this approach even to reproduce Eq. (23) for a round current loop.) From the deep analogy with electrostatics, we may expect that in this case we could recover the field from the solution of a certain partial boundary problem for the field's potential, in this case the vector-potential **A** defined by Eq. (28). However, despite the similarity of this formula and Eq. (1.38) for ϕ, that was emphasized above, there are two additional issues we should tackle in the magnetic case.

First, finding vector-potential distribution means determining three scalar functions (say, A_x, A_y, and A_z), rather than one (ϕ). Second, generally the differential equation satisfied by **A** is more complex than the Poisson equation for ϕ. Indeed, plugging Eq. (27) into Eq. (35), we get

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{j} \,. \tag{5.42}$$

If we wrote the left-hand part of this equation in (say, Cartesian) components, we would see that they are much more interwoven than in the Laplace operator, and hence much less convenient for using the orthogonal coordinate approach or the variable separation method. In order to remedy the situation, let us apply to Eq. (42) the now-familiar identity (31). The result is

[15] Applying the Ampère law to a circular contour of radius ρ, coaxial with the solenoid, we see that the field outside (but not inside!) it has an azimuthal component B_φ, similar to that of the straight wire (see Eq. (38) above) and hence (at $N >> 1$) much weaker than the longitudinal field inside the solenoid – see Eq. (40).

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j}. \qquad (5.43)$$

We see that if we could kill the first term in the left-hand part, for example if $\nabla\cdot\mathbf{A} = 0$, the second term would give us a set of independent Poisson equations for each Cartesian component of vector **A**.

In this context, let us discuss what discretion do we have in the potentials' choice. In electrostatics, we might add to any function ϕ', that satisfies Eq. (1.33), not only an arbitrary constant, but also an arbitrary function of time, without affecting the electric field:

$$-\nabla\left[\phi' + f(t)\right] = -\nabla\phi' = \mathbf{E}. \qquad (5.44)$$

Similarly, using the fact that curl of the gradient of any scalar function equals zero,[16] we may add to any function **A**', that satisfies Eq. (27) for a given field **B**, not only a constant, but even a gradient of an arbitrary function $\chi(\mathbf{r}, t)$, because

$$\nabla \times (\mathbf{A}' + \nabla\chi) = \nabla \times \mathbf{A}' + \nabla \times (\nabla\chi) = \nabla \times \mathbf{A}' = \mathbf{B}. \qquad (5.45)$$

Such additions, keeping the actual (observable) fields intact, are called *gauge transformations*.[17] Let us see what such a transformation does to $\nabla\cdot\mathbf{A}'$:

$$\nabla \cdot (\mathbf{A}' + \nabla\chi) = \nabla \cdot \mathbf{A}' + \nabla^2\chi. \qquad (5.46)$$

Hence we can always choose a function χ in such a way that the right-hand part of this relation, and hence the divergence of the transformed vector-potential, $\mathbf{A} \equiv \mathbf{A}' + \nabla\chi$, would vanish at all points. In this case, Eq. (43) is reduced to the vector Poisson equation

Poisson equation for **A**

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j}, \qquad (5.47)$$

As we know from the electrostatics (please compare Eqs. (1.38) and (1.41)), the vector-potential defined by Eq. (28) does satisfy this equation. So, the so-called *Coulomb gauge* condition,

Coulomb gauge

$$\nabla \cdot \mathbf{A} = 0, \qquad (5.48)$$

reduces the set of the functions $\mathbf{A}(\mathbf{r})$ that satisfy Eq. (27) to the actual vector-potential (28). However, this condition still leaves some liberty in the vector-potential selection. In order to demonstrate that, and also to get a better feeling of vector-potential's distribution in space, let us calculate it for two important particular cases.

First, let us revisit the straight wire problem (Fig. 5). As Eq. (28) shows, in this case vector **A** has just one component (along the axis z). Moreover, due to the problem's axial symmetry, its magnitude may only depend on the distance from the axis: $\mathbf{A} = \mathbf{n}_z A(\rho)$. Hence, the gradient of **A** is directed across axis z, so that Eq. (48) is satisfied even for this vector, i.e. the Poisson equation (47) is satisfied even for the original vector **A**. For our symmetry ($\partial/\partial\varphi = \partial/\partial z = 0$), the Laplace operator, written in cylindrical coordinates, has just one term,[18] reducing Eq. (47) to

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dA}{d\rho}\right) = -\mu_0 j(\rho). \qquad (5.49)$$

[16] See, e.g., MA Eq. (11.1).
[17] The use of term "gauge" (originally meaning "a measure" or "a scale") in this context is purely historic, so the reader should not try to find too much hidden sense in it.
[18] See, e.g., MA Eq. (10.3).

Multiplying both parts of this equation by ρ and integrating them over the coordinate once, we get

$$\rho \frac{dA}{d\rho} = -\mu_0 \int_0^{\rho} j(\rho')\rho' d\rho' + \text{const}. \tag{5.50}$$

Since in the cylindrical coordinates, for our symmetry,[19] $B = - dA/d\rho$, Eq. (50) is nothing else than our old result (38) for the magnetic field.[20] However, let us continue the integration, at least for the region outside the wire, where the function $A(\rho)$ depends only on the full current I rather than on the current distribution inside the wire. Dividing both parts of Eq. (50) by ρ, and integrating them over that coordinate again, we get

$$A(\rho) = -\frac{\mu_0 I}{2\pi} \ln \rho + \text{const}, \quad \text{where } I = 2\pi \int_0^R j(\rho)\rho d\rho. \tag{5.51}$$

As a reminder, we had the similar logarithmic behavior for the electrostatic potential outside a uniformly charged straight line. This is natural, because the Poisson equations for both cases are similar.

Now let us find the vector-potential for the long solenoid (Fig. 6a), with its uniform magnetic field. Since Eq. (28) prescribes vector **A** to follow the direction of the current, we can start with looking for it in the form $\mathbf{A} = \mathbf{n}_\varphi A(\rho)$. (This is especially natural if the solenoid's cross-section is circular.) With this orientation of **A**, the same general expression for the curl operator in cylindrical coordinates yields $\nabla \times \mathbf{A} = \mathbf{n}_z (1/\rho) d(\rho A)/d\rho$. According to the definition (27) of **A**, this expression should be equal to **B,** in our case equal to $\mathbf{n}_z B$, with constant B – see Eq. (40). Integrating this equality, and selecting such integration constant so that $A(0)$ is finite, we get

$$A(\rho) = \frac{B\rho}{2}. \tag{5.52}$$

Plugging this result into the general expression for the Laplace operator in the cylindrical coordinates,[21] we see that the Poisson equation (47) with $\mathbf{j} = 0$ (i.e. the Laplace equation), is satisfied again – which is natural since for this distribution, $\nabla \cdot \mathbf{A} = 0$. However, Eq. (52) is not the unique (or even the simplest) solution of the problem. Indeed, using the well-known expression for the curl operator in Cartesian coordinates,[22] it is straightforward to check that either function $\mathbf{A}' = \mathbf{n}_y Bx$, or function $\mathbf{A}'' = -\mathbf{n}_x By$, or any of their weighed sums, for example $\mathbf{A}''' = (\mathbf{A}' + \mathbf{A}'')/2 = B(-\mathbf{n}_x y + \mathbf{n}_y x)/2$, also give the same magnetic field, and also evidently satisfy the Laplace equation. If such solutions do not look very natural due to their anisotropy in the [x, y] plane, please consider the fact that they represent the uniform magnetic field regardless of its source (e.g., of the shape of long solenoid's cross-section). Such choices of vector-potential may be very convenient for some problems, for example for the analysis of the 2D motion of a charged quantum particle in the perpendicular magnetic field, giving the famous Landau energy levels.[23]

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[19] See, e.g., MA Eq. (10.5) with $\partial/\partial\varphi = \partial/\partial z = 0$.

[20] Since the magnetic field at the wire axis has to be zero (otherwise, being perpendicular to the axis, where would it be directed?), the integration constant in Eq. (50) should be zero.

[21] See, e.g., MA Eq. (10.6).

[22] See, e.g., MA Eq. (8.5).

[23] See, e.g., QM Sec. 3.2.

5.3. Magnetic energy, flux, and inductance

Considering currents flowing in a system as generalized coordinates, magnetic forces (1) between them are their unique functions, and in this sense the magnetic interaction energy U may be considered a potential energy of the system. The apparent (but deceptive) way to guess the energy is to use the analogy between Eq. (1) and its electrostatic analog, Eq. (2). As we know from Chapter 1, if these densities describe the distribution of the same charge, i.e. if $\rho'(\mathbf{r}) = \rho(\mathbf{r})$, then the self-interaction of its elementary components correspond to the potential energy expressed by Eq. (1.61):

$$U = \frac{1}{4\pi\varepsilon_0}\frac{1}{2}\int d^3r \int d^3r' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}. \qquad (5.53)$$

Using the analogy, for the magnetic interaction between elementary components of the same current, with density $\mathbf{j}(\mathbf{r}) = \mathbf{j}'(\mathbf{r})$, we could guess that

Magnetic interaction energy

$$U = \frac{\mu_0}{4\pi}\frac{1}{2}\int d^3r \int d^3r' \frac{\mathbf{j}(\mathbf{r})\cdot\mathbf{j}(\mathbf{r}')}{\left|\mathbf{r}-\mathbf{r}'\right|}, \qquad (5.53)$$

while for independent currents the coefficient ½ should be removed. Now let me confess that this is a *wrong* way to get this *correct* result. Indeed, the sign in Eq. (1) is opposite to that in Eq. (2), so that following this argumentation we would get Eq. (53) with the minus sign. The reason of this paradox is fundamental: fixing electric charges does not require external interference (work), while the maintenance of currents generally does. Strictly speaking, a derivation of Eq. (53) required additional experimental fact, the *Faraday induction law*. However, I would like to defer its discussion until the beginning of the next chapter, and for now ask the reader to believe me that the sign in Eq. (53) is correct.

Due to the importance of this relation, let us rewrite it in several other forms, beneficial for different applications. First of all, just as in electrostatics, Eq. (54) may be recast into a potential-based form. Indeed, using definition (28) of the vector-potential $\mathbf{A}(\mathbf{r})$, Eq. (54) becomes[24]

$$U = \frac{1}{2}\int \mathbf{j}(\mathbf{r})\cdot\mathbf{A}(\mathbf{r})d^3r. \qquad (5.55)$$

This formula, that is a clear magnetic analog of Eq. (1.62) of electrostatics, is very popular among theoretical physicists, because it is very handy for the field theory manipulations. However, for many calculations it is more convenient to have a direct expression of energy via the magnetic field. Again, this may be done very similarly to what we have done in Sec. 1.3 for electrostatics, i.e. plugging into Eq. (55) the current density expressed from Eq. (35) to transform it as[25]

$$U = \frac{1}{2}\int \mathbf{j}\cdot\mathbf{A}d^3r = \frac{1}{2\mu_0}\int \mathbf{A}\cdot(\nabla\times\mathbf{B})d^3r = \frac{1}{2\mu_0}\int \mathbf{B}\cdot(\nabla\times\mathbf{A})d^3r - \frac{1}{2\mu_0}\int \nabla\cdot(\mathbf{A}\times\mathbf{B})d^3r. \qquad (5.56)$$

Now using the divergence theorem, the second integral may be transformed into a surface integral of product $(\mathbf{A}\times\mathbf{B})_n$. Equations (27)-(28) show that if the current distribution $\mathbf{j}(\mathbf{r})$ is localized, this product drops with distance r faster than $1/r^2$, so that if the integration volume is large enough, the surface

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[24] This relation remains the same in the Gaussian units, because in those units both Eq. (28) and Eq. (54) should be stripped of their $\mu_0/4\pi$ coefficients.

[25] For that, we may use MA Eq. (11.7) with $\mathbf{f} = \mathbf{A}$ and $\mathbf{g} = \mathbf{B}$, giving $\mathbf{A}\cdot(\nabla\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \nabla\cdot(\mathbf{A}\times\mathbf{B})$.

integral is negligible. In the remaining first integral, we may use Eq. (27) to recast $\nabla\times\mathbf{A}$ into the magnetic field. As a result, we get a very simple and fundamental formula.

$$U = \frac{1}{2\mu_0}\int B^2 d^3r . \tag{5.57a}$$

Just as with the electric field, this expression may be interpreted as a volume integral of the *magnetic energy density u*:

Magnetic field energy

$$U = \int u(\mathbf{r})d^3r, \quad \text{with } u(\mathbf{r}) \equiv \frac{1}{2\mu_0}\mathbf{B}^2(\mathbf{r}), \tag{5.57b}$$

clearly similar to Eq. (1.67).[26] Again, the conceptual choice between the spatial localization of magnetic energy – either at the location of electric currents only, as implied by Eqs. (54) and (55), or in all regions where the magnetic field exists, as apparent from Eq. (57b), cannot be done within the framework of magnetostatics, and only electrodynamics gives the decisive preference for the latter choice.

For the practically important case of currents flowing in several thin wires, Eq. (54) may be first integrated over the cross-section of each wire, just as was done at the derivation of Eq. (4). Again, since the integral of the current density over k^{th} wire's cross-section is just the current I_k in the wire, and cannot change along its length, it may be taken from the remaining integrals, giving

$$U = \frac{\mu_0}{4\pi}\frac{1}{2}\sum_{k,k'} I_k I_{k'} \oint_{l_k}\oint_{l_{k'}} \frac{d\mathbf{r}_k \cdot d\mathbf{r}_{k'}}{|\mathbf{r}_k - \mathbf{r}_{k'}|}, \tag{5.58}$$

where l is the full length of the wire loop. Note that Eq. (58) is valid if currents I_k are independent of each other, because the double sum counts each current pair twice, compensating coefficient ½ in front of the sum. It is useful to decompose this relation as

$$U = \frac{1}{2}\sum_{k,k'} I_k I_{k'} L_{kk'}, \tag{5.59}$$

Mutual inductance coefficients

$$L_{kk'} \equiv \frac{\mu_0}{4\pi}\oint_{l_k}\oint_{l_{k'}} \frac{d\mathbf{r}_k \cdot d\mathbf{r}_{k'}}{|\mathbf{r}_k - \mathbf{r}_{k'}|}, \tag{5.60}$$

Coefficient $L_{kk'}$ in the quadratic form (59), with $k \neq k'$, is called the *mutual inductance* between current loops k and k', while the diagonal coefficient $L_k \equiv L_{kk}$ is called the *self-inductance* (or just *inductance*) of k^{th} loop.[27] From the symmetry of Eq. (60) with respect to the index swap, $k \leftrightarrow k'$, it evident that the matrix of coefficients $L_{kk'}$ is symmetric:[28]

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[26] The transfer to the Gaussian units in Eqs. (77)-(78) may be accomplished by the usual replacement $\mu_0 \to 4\pi$, thus giving, in particular, $u = B^2/8\pi$.

[27] As evident from Eq. (60), these coefficients depend only on the geometry of the system. Moreover, in the Gaussian units, in which Eq. (60) is valid without the factor $\mu_0/4\pi$, the inductance coefficients have the dimension of length (centimeters). The SI unit of inductance is called the *henry*, abbreviated H - after J. Henry, 1797-1878, who in particular discovered the effect of electromagnetic induction (see Sec. 6.1) independently of M. Faraday.

[28] Note that the matrix of the mutual inductances $L_{jj'}$ is very much similar to the matrix of *reciprocal* capacitance coefficients $p_{kk'}$ – for example, compare Eq. (62) with Eq. (2.21).

$$L_{kk'} = L_{k'k}, \tag{5.61}$$

so that for the practically important case of two interacting currents I_1 and I_2, Eq. (59) reads

$$U = \frac{1}{2}L_1 I_1^2 + M I_1 I_2 + \frac{1}{2}L_2 I_2^2, \tag{5.62}$$

where $M \equiv L_{12} = L_{21}$ is the mutual inductance coefficient.

These formulas clearly show the importance of self- and mutual inductances, so I will demonstrate their calculation for at least a few basic geometries. Before doing that, however, let me recast Eq. (58) into one more form that may facilitate such calculations. Namely, let us notice that for the magnetic field induced by current I_k in a thin wire, Eq. (28) is reduced to

$$\mathbf{A}_k(\mathbf{r}) = \frac{\mu_0}{4\pi} I_k \int_{l'} \frac{d\mathbf{r}_k}{|\mathbf{r} - \mathbf{r}_k|}, \tag{5.63}$$

so that Eq. (58) may be rewritten as

$$U = \frac{1}{2}\sum_{k,k'} I_k \oint_{l_k} \mathbf{A}_{k'}(\mathbf{r}_k)\cdot d\mathbf{r}_{k'} . \tag{5.64}$$

But according to the same Stokes theorem that was used earlier in this chapter to derive the Ampère law, and Eq. (27), such integral is nothing more than the *magnetic field flux* (more frequently called just the *magnetic flux*) through a surface S limited by the contour l :[29]

Magnetic flux

$$\oint_l \mathbf{A}(\mathbf{r})\cdot d\mathbf{r} = \int_S (\nabla \times A)_n d^2 r = \int_S B_n d^2 r \equiv \Phi . \tag{5.65}$$

As a result, Eq. (64) may be rewritten as

$$U = \frac{1}{2}\sum_{k,k'} I_k \Phi_{kk'}, \tag{5.66}$$

where $\Phi_{kk'}$ is the flux of the field induced by k'-th current through the loop of the k-th current. Comparing this expression with Eq. (59), we see that

Magnetic flux from currents

$$\Phi_{kk'} \equiv \int_{S_k} (\mathbf{B}_{k'})_n d^2 r = L_{kk'} I_{k'}, \tag{5.67}$$

This expression not only gives us one more means for calculating coefficients $L_{kk'}$, but also shows their physical sense: the mutual inductance characterizes how much field (colloquially, "how many field lines") induced by current $I_{k'}$ penetrate the loop of current I_k, and vice versa. Since due to the linear superposition principle, the total flux piercing k-th loop may be presented as

$$\Phi_k \equiv \sum_{k'} \Phi_{kk'} = \sum_{k'} L_{kk'} I_{k'} . \tag{5.68}$$

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[29] The SI unit of magnetic flux is called *weber*, abbreviated Wb - after W. Weber, who in particular co-invented (with C. Gauss) the electromagnetic telegraph, and in 1856 was first, together with R. Kohlrausch, to notice that the value of (in modern terms) $1/(\varepsilon_0\mu_0)^{1/2}$, derived from electrostatic and magnetostatic measurements, coincides with the independently measured speed of light c, giving an important motivation for Maxwell's theory.

For example, for the system of two currents this expression is reduced to a clear analog of Eqs. (2.19):

$$\Phi_1 = L_1 I_1 + M I_2, \qquad \Phi_2 = M I_1 + L_2 I_2 . \tag{5.69}$$

For the even simpler case of a single current,

Φ and *U* of a single current

$$\Phi = LI, \tag{5.70}$$

so that the magnetic energy of the current may be presented in several equivalent forms:

$$U = \frac{L}{2} I^2 = \frac{1}{2} I\Phi = \frac{1}{2L}\Phi^2 . \tag{5.71}$$

These relations, similar to Eqs. (2.14)-(2.15) of electrostatics, show that the self-inductance L of a current loop may be considered as a measure of system's magnetic energy at fixed current.

Now we are well equipped for the calculation of inductances, having three options. The first one is to use Eq. (60) directly.[30] The second one is to calculate the magnetic field energy from Eq. (57) as the function of currents I_k in the system, and then use Eq. (59) to find all coefficients $L_{kk'}$. For example, for a system with just one current, Eq. (71) yields

$$L = \frac{U}{I^2/2} . \tag{5.72}$$

Finally, if the system consists of thin wires, so that the loop areas S_k and hence fluxes $\Phi_{kk'}$ are well defined, we may calculate them from Eq. (65), and then use Eq. (67) to find the inductances.

Actually, the first two options may have advantages over the third one even for such system of thin wires for whom the notion of magnetic flux is not quite clear. As an important example, let us find inductance of a long solenoid - see Fig. 6a. We have already calculated the magnetic field inside it – see Eq. (40) - so that, due to the field uniformity, the magnetic flux piercing each wire turn is just

$$\Phi_1 = BA = \mu_0 n I A, \tag{5.73}$$

where A is the area of solenoid's cross-section - for example πR^2 for a round solenoid, though Eq. (40) is more general. Comparing Eqs. (73) and (67), one might wrongly conclude that $L = \Phi_1/I = \mu_0 nA$ [WRONG!], i.e. that the solenoid's inductance is independent on its length. Actually, the magnetic flux Φ_1 pierces *each* wire turn, so that the total flux through the *whole* current loop, consisting of N turns, is

$$\Phi = N\Phi_1 = \mu_0 n^2 l A I, \tag{5.74}$$

and the correct expression for solenoid's inductance is

$$L = \frac{\Phi}{I} = \mu_0 n^2 l A, \tag{5.75}$$

i.e. the inductance per unit length is constant: $L/l = \mu_0 n^2 A$. Since this reasoning may seem a bit flimsy, it is prudent to verify it by using Eq. (72) to calculate the full magnetic energy inside the solenoid (neglecting minor fringe and external field contributions):

[30] Numerous applications of this *Neumann formula* to electrical engineering problems may be found, for example, in the classical text F. Grover, *Inductance Calculations*, Dover, 1946.

$$U = \frac{1}{2\mu_0} B^2 A l = \frac{1}{2\mu_0} (\mu_0 n I)^2 A l = \mu_0 n^2 l A \frac{I^2}{2}. \quad (5.76)$$

Plugging this result into Eq. (72) immediately confirms result (75).

The use of the first two options for inductance calculation becomes inevitable for continuously distributed currents. As an example, let us calculate self-inductance L of a long coaxial cable with the cross-section shown in the Fig. 7.[31]

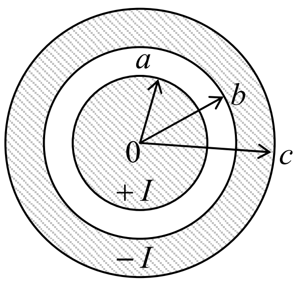

Fig. 5.7. Cross-section of a coaxial cable.

Let us assume that the current is uniformly distributed over the cross-sections of both conductors. (As we know from the previous chapter, such distribution indeed takes place if both the internal and external conductors are made of a uniform resistive material.) First, we should calculate the radial distribution of the magnetic field (that of course has only one, azimuthal component, because of the axial symmetry of the problem). This distribution may be immediately found from the application of the Ampère law to circles of radii ρ within four different ranges:

$$2\pi\rho B = \mu_0 I\Big|_{\text{piercing the circle area}} = \mu_0 I \times \begin{cases} \dfrac{\rho^2}{a^2}, & \text{for } \rho < a, \\ 1, & \text{for } a < \rho < b, \\ \dfrac{c^2 - \rho^2}{c^2 - b^2}, & \text{for } b < \rho < c, \\ 0, & \text{for } c < \rho. \end{cases} \quad (5.77)$$

Now, an elementary integration yields the magnetic energy per unit length of the cable:

$$\begin{aligned} \frac{U}{l} &= \frac{1}{2\mu_0} \int B^2 d^2 r = \frac{\pi}{\mu_0} \int_0^\infty B^2 \rho d\rho = \frac{\mu_0 I^2}{4\pi} \left[\int_0^a \left(\frac{\rho}{a^2} \right)^2 \rho d\rho + \int_a^b \left(\frac{1}{\rho} \right)^2 \rho d\rho + \int_b^c \left(\frac{c^2 - \rho^2}{\rho(c^2 - b^2)} \right)^2 \rho d\rho \right] \\ &= \frac{\mu_0}{2\pi} \left[\ln\frac{b}{a} + \frac{c^2}{c^2 - b^2} \left(\frac{c^2}{c^2 - b^2} \ln\frac{c}{b} - \frac{1}{2} \right) \right] \frac{I^2}{2}. \end{aligned} \quad (5.78)$$

From here, and Eq. (72), we get the final answer:

$$\frac{L}{l} = \frac{\mu_0}{2\pi} \left[\ln\frac{b}{a} + \frac{c^2}{c^2 - b^2} \left(\frac{c^2}{c^2 - b^2} \ln\frac{c}{b} - \frac{1}{2} \right) \right]. \quad (5.79)$$

[31] As a reminder, the mutual capacitance C between the conductors of such a system was calculated in Sec. 2.3. As will be discussed in Chapter 7 below, the pair of parameters L and C define the propagation of the most important, TEM mode of electromagnetic waves along the cable.

Note that for the particular case of a thin outer conductor, $c - b << b$, this expression reduces to

$$\frac{L}{l} \approx \frac{\mu_0}{2\pi}\left(\ln\frac{b}{a} + \frac{1}{4}\right), \tag{5.80}$$

where the first term in the parentheses may be traced back to the contribution of the magnetic field energy in the free space between the conductors. This distinction is important for some applications, because in superconductor cables, as well as resistive-metal cables as high frequencies (to be discussed in the next chapter), the field does not penetrate the conductor bulk, so that Eq. (80) is valid without the last term, 1/4, in the parentheses, which is due to the magnetic field energy inside the wire.

As the last example, let us calculate the mutual inductance between a long straight wire and a round wire loop adjacent to it (Fig. 8), neglecting the thickness of both wires.

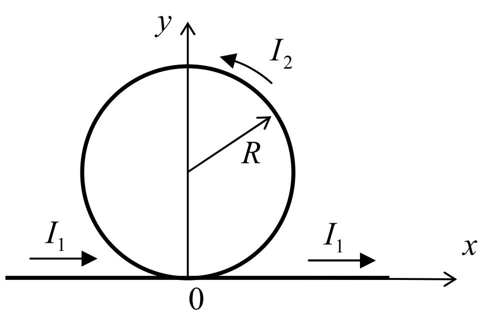

Fig. 5.8. Study case for the mutual inductance calculation.

Here there is no problem with using the last formalism, based on the magnetic flux calculation. Indeed, in the Cartesian coordinates shown in Fig. 8, Eq. (20) reads $B_1 = \mu_0 I_1/2\pi y$, giving the following magnetic flux through the round wire loop:

$$\Phi_{21} = \frac{\mu_0 I_1}{2\pi}\int_{-R}^{+R} dx \int_{R-(R^2-x^2)^{1/2}}^{R+(R^2-x^2)^{1/2}} dy \frac{1}{y} = \frac{\mu_0 I_1}{\pi}\int_0^R \ln\frac{R+(R^2-x^2)^{1/2}}{R-(R^2-x^2)^{1/2}} dx = \frac{\mu_0 I_1 R}{\pi}\int_0^1 \ln\frac{1+(1-\xi^2)^{1/2}}{1+(1-\xi^2)^{1/2}} d\xi. \tag{5.81}$$

This is a table integral equal to π,[32] so that $\Phi_{21} = \mu_0 I_1 R$, and the final answer for the mutual inductance $M = L_{12} = L_{21} = \Phi_{21}/I_1$ is finite (and very simple):

$$M = \mu_0 R, \tag{5.82}$$

despite magnetic field's divergence at the lowest point of the loop ($y = 0$). Note that in contrast with the finite *mutual* inductance of this system, *self*-inductances of both wires are formally infinite in the thin-wire limit – see, e.g., Eq. (80), that in the limit $b/a >> 1$ describes a thin straight wire. However, since this divergence is very weak (logarithmic), it is quenched by any deviation from this perfect geometry. For example, a good estimate of the inductance of a wire of a large but finite length l may be obtained from Eq. (81) via the replacement of b with l:

$$L \sim \frac{\mu_0}{2\pi} l \ln\frac{l}{a}. \tag{5.83}$$

[32] See, e.g., MA Eq. (6.13) for $a = 1$.

(Note, however, that the exact result depends on where from/to the current flows beyond that segment.) A close estimate, with l replaced with $2\pi R$, and b replaced with R, is valid for the self-inductance of the round loop. A more exact calculation of this inductance, which would be asymptotically correct in the limit $a << R$, is a very useful exercise, which is highly recommended to the reader. [33]

5.4. Magnetic dipole moment, and magnetic dipole media

The most natural way of description of magnetic media parallels that described in Chapter 3 for dielectrics, and is based on properties of *magnetic dipoles*. To introduce this notion quantitatively, let us consider, just as in Sec. 3.1, a spatially-localized system with current distribution $\mathbf{j}(\mathbf{r})$, whose magnetic field is measured at relatively large distances $r >> r'$ (Fig. 9).

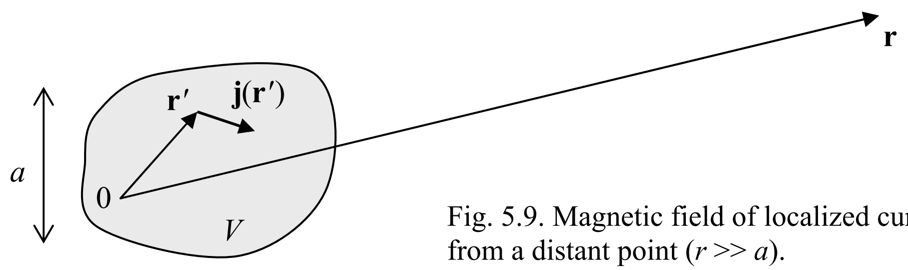

Fig. 5.9. Magnetic field of localized currents, observed from a distant point ($r >> a$).

Applying the truncated Taylor expansion (3.4) to definition (28) of the vector potential, we get

$$\mathbf{A}(\mathbf{r}) \approx \frac{\mu_0}{4\pi}\left[\frac{1}{r}\int_V \mathbf{j}(\mathbf{r}')d^3r' + \frac{1}{r^3}\int_V (\mathbf{r}\cdot\mathbf{r}')\mathbf{j}(\mathbf{r}')d^3r'\right]. \tag{5.84}$$

Due to the vector character of this potential, we have to depart slightly from the approach of Sec. 3.1 and use the following vector algebra identity:[34]

$$\int_V \left[f(\mathbf{j}\cdot\nabla g) + g(\mathbf{j}\cdot\nabla f)\right] d^3r = 0 \tag{5.85}$$

that is valid for any pair of smooth (differentiable) scalar functions $f(\mathbf{r})$ and $g(\mathbf{r})$, and any vector function $\mathbf{j}(\mathbf{r})$ that, as the dc current density, satisfies the continuity condition $\nabla\cdot\mathbf{j} = 0$ and whose normal component vanishes on its surface.

First, let us use Eq. (85) with $f \equiv 1$ and g equal to any component of the radius-vector $\mathbf{r}$: $g = r_i$ (i = 1, 2, 3). Then it yields

$$\int_V (\mathbf{j}\cdot\mathbf{n}_i)d^3r = \int_V j_i d^3r = 0, \tag{5.86}$$

so that for the vector as the whole

[33] Its solution may be found, for example, just after Sec. 34 of L. Landau et al., *Electrodynamics of Continuous Media*, 2nd ed., Butterwort Heinemann, 1984.

[34] See, e.g., MA Eq. (12.3) with additional condition $j_n\big|_S = 0$, pertinent for space-restricted currents.

$$\int_V \mathbf{j}(\mathbf{r}) d^3r = 0 , \tag{5.87}$$

showing that the first term in the right-hand part of Eq. (84) equals zero. Next, let us use Eq. (85) with $f = r_i$, $g = r_{i'}$ ($i, i' = 1, 2, 3$); then it yields

$$\int_V \left(r_i j_{i'} + r_{i'} j_i \right) d^3r = 0 , \tag{5.88}$$

so that the i^{th} Cartesian component of the second integral in Eq. (84) may be transformed as

$$\begin{aligned} \int_V (\mathbf{r}\cdot\mathbf{r}') j_i d^3r' &= \int_V \sum_{i'=1}^{3} r_{i'} r'_{i'} j_i d^3r' = \frac{1}{2}\sum_{i'=1}^{3} r_{i'} \int_V (r'_{i'} j_i + r'_{i'} j_i) d^3r' \\ &= \frac{1}{2}\sum_{i'=1}^{3} r_{i'} \int_V (r'_{i'} j_i - r'_{i'} j_{i'}) d^3r' = -\frac{1}{2}\left[\mathbf{r} \times \int_V (\mathbf{r}' \times \mathbf{j}) d^3r' \right]_i . \end{aligned} \tag{5.89}$$

As a result, Eq. (85) may be rewritten as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m}\times\mathbf{r}}{r^3} , \tag{5.90}$$

Magnetic dipole and its potential

where vector **m**, defined as[35]

$$\mathbf{m} \equiv \frac{1}{2} \int_V \mathbf{r} \times \mathbf{j}(\mathbf{r})\, d^3r , \tag{5.91}$$

is called the *magnetic dipole moment* of our system - that itself, within approximation (90), is called the *magnetic dipole*.

Note a close analogy between **m** and the angular momentum of a non-relativistic particle with mass m_k:

$$\mathbf{L}_k \equiv \mathbf{r}_k \times \mathbf{p}_k = \mathbf{r}_k \times m_k \mathbf{v}_k , \tag{5.92}$$

where $\mathbf{p}_k = m_k \mathbf{v}_k$ is its mechanical momentum. Indeed, for a continuum of such particles with the same electric charge q, with the spatial density n, $\mathbf{j} = qn\mathbf{v}$, and Eq. (91) yields

$$\mathbf{m} = \int_V \frac{1}{2} \mathbf{r} \times \mathbf{j}\, d^3r = \int_V \frac{nq}{2} \mathbf{r} \times \mathbf{v}\, d^3r , \tag{5.93}$$

while the total angular momentum of such continuous system of particles of the same mass ($m_k = m_0$) is

$$\mathbf{L} = \int_V n m_0 \mathbf{r} \times \mathbf{v} d^3r ,$$

so that we get a very straightforward relation

m vs. **L**

$$\mathbf{m} = \frac{q}{2m_0} \mathbf{L} . \tag{5.95}$$

[35] In the Gaussian units, definition (91) is kept valid, so that Eq. (90) is stripped of the factor $\mu_0/4\pi$.

For the orbital motion, this classical relation survives in quantum mechanics for operators and hence for eigenvalues, in whom the angular momentum is quantized in the units of the Plank's constant $\hbar$, so that for an electron, the orbital magnetic moment is always a multiple of the so-called *Bohr magneton*

$$\mu_B \equiv \frac{e\hbar}{2m_e}, \tag{5.96}$$

Bohr magneton

where m_e is the free electron mass.[36] However, for particles with spin, such a universal relation between vectors **m** and **L** is no longer valid. For example, electron's spin $s = ½$ gives contribution $\hbar/2$ to the mechanical momentum, but its contribution to the magnetic moment it still very close to μ_B.[37]

The next important example of a magnetic dipole is a *planar* wire loop limiting area A (of an arbitrary shape), carrying current I, for which **m** has a surprisingly simple form,

$$\mathbf{m} = I\mathbf{A}, \tag{5.97}$$

where the modulus of vector **A** equals area A, and its direction is perpendicular to loop's plane. This formula may be readily proved by noticing that if we select the coordinate origin on the plane of the loop (Fig. 10), then the elementary component of the magnitude of integral (91),

$$m = \frac{1}{2}\left|\oint_C \mathbf{r}\times I d\mathbf{r}\right| = I\oint_C \left|\frac{1}{2}\mathbf{r}\times d\mathbf{r}\right| = I\oint_C \frac{1}{2} r^2 d\varphi, \tag{5.98}$$

is just the elementary area $dA = (1/2)rdh = (1/2)rd(r\sin\varphi) = r^2 d\varphi/2$.

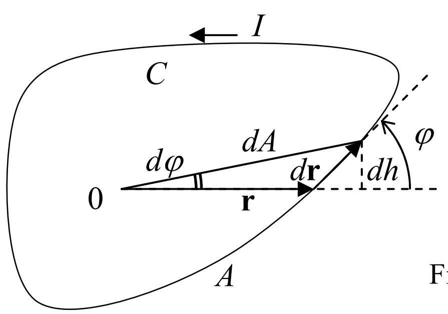

Fig. 5.10. Planar current loop.

The combination of Eqs. (96) and (97) allows a useful estimate of the scale of atomic currents, by finding what current I should flow in a circular loop of atomic size scale (the Bohr radius) $r_B \approx 0.5\times10^{-10}$ m, i.e. of area $A \approx 10^{-20}$ m^2, to produce a magnetic moment equal to μ_B.[38] The result is surprisingly macroscopic: $I \sim 1$ mA (quite comparable to the currents driving your earbuds :-). Though this estimate should not be taken too literally, due to the quantum-mechanical spread of electron's wavefunctions, it is very useful for getting a feeling how significant the atomic magnetism is and hence why ferromagnets may provide such a strong field.

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[36] In SI units, $m_e \approx 0.91\times10^{-30}$ kg, so that $\mu_B \approx 0.93\times10^{-23}$ J/T.

[37] See, e.g., QM Sec. 4.1 and beyond.

[38] Another way to arrive at the same estimate is to take $I \sim ef = e\omega/2\pi$ with $\omega \sim 10^{16}$ s^{-1} being the typical frequency of radiation due to atomic interlevel quantum transitions.

After these illustrations, let us return to Eq. (90). Plugging it into the general formula (27), we may calculate the magnetic field of a magnetic dipole:

Magnetic dipole's field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3\mathbf{r}(\mathbf{r}\cdot\mathbf{m}) - \mathbf{m}r^2}{r^5}. \qquad (5.99)$$

The structure of this formula *exactly* duplicates that of Eq. (3.15) for the electric dipole field. Because of this similarity, the energy of a dipole in an external field, and hence the torque and force exerted on it by the field, are also absolutely similar to the expressions for an electric dipole - see Eqs. (3.15)-(3.18):

Magnetic dipole in external field

$$U = -\mathbf{m}\cdot\mathbf{B}_{\text{ext}}, \qquad (5.100)$$

and as a result,

$$\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}_{\text{ext}}, \qquad (5.101)$$

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}_{\text{ext}}). \qquad (5.102)$$

Now let us consider a system of many magnetic dipoles (e.g., atoms or molecules), distributed in space with density n. Then we can use Eq. (90) (generalized in the evident way for an arbitrary position, $\mathbf{r}'$, of a dipole), and the linear superposition principle, to calculate the "macroscopic" component of the vector-potential $\mathbf{A}$ - in other words, dipole's potential averaged over short-scale variations on the inter-dipole distances:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3r', \qquad (5.103)$$

where $\mathbf{M} \equiv n\mathbf{m}$ is the *macroscopic* (average) *magnetization*, i.e. the magnetic moment per unit volume. Transforming this integral absolutely similarly to how Eq. (3.27) had been transformed into Eq. (3.29), we get:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3r'. \qquad (5.104)$$

Comparing this result with Eq. (28), we see that $\nabla\times\mathbf{M}$ is equivalent, in its effect, to the density $\mathbf{j}_{\text{ef}}$ of a certain effective "magnetization current". Just as the electric-polarization "charge" ρ_{ef} discussed in Sec. 3.2 (see Fig. 3.3), $\mathbf{j}_{\text{ef}} = \nabla\times\mathbf{M}$ may be interpreted the uncompensated part of vortex currents representing single magnetic dipoles (Fig. 11).

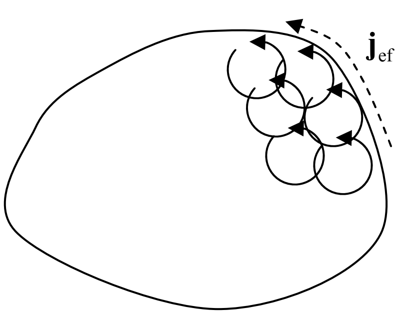

Fig. 5.11. Cartoon illustrating the physical nature of the "magnetization current" $\mathbf{j}_{\text{ef}} = \nabla\times\mathbf{M}$.

Now, using Eq. (28) to add the possible contribution from "stand-alone" currents **j**, not included into the currents of microscopic dipoles, we get the general equation for the vector-potential of the macroscopic field:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{[\mathbf{j}(\mathbf{r}') + \nabla' \times \mathbf{M}(\mathbf{r}')]}{|\mathbf{r} - \mathbf{r}'|} d^3r' . \tag{5.105}$$

Repeating the calculations that have led us from Eq. (28) to the Maxwell equation (35), with the account of the magnetization current term, for the macroscopic magnetic field **B** we get[39]

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{j} + \nabla \times \mathbf{M}). \tag{5.106}$$

Following the same philosophy as in Sec. 3.2, we may recast this equation as

$$\nabla \times \mathbf{H} = \mathbf{j}, \tag{5.107}$$

where a new field defined as

$$\mathbf{H} \equiv \frac{\mathbf{B}}{\mu_0} - \mathbf{M}, \tag{5.108}$$

Magnetic field **H**

by historic reasons (and very unfortunately) is also called the *magnetic field*.[40] It is crucial to remember that the physical sense of field **H** is very much different from field **B**. In order to understand the difference better, let us use Eq. (107) to complete a macroscopic analog of system (36), called the *macroscopic Maxwell equations* (again, so far for the stationary case $\partial/\partial t = 0$):

$$\begin{aligned} &\nabla \times \mathbf{E} = 0, \qquad \nabla \times \mathbf{H} = \mathbf{j}, \\ &\nabla \cdot \mathbf{D} = \rho, \qquad \nabla \cdot \mathbf{B} = 0. \end{aligned} \tag{5.109}$$

Stationary macroscopic Maxwell equations

One can clearly see that the roles of vector fields **D** and **H** are very similar: they could be called "would-be" fields - which *would be* induced by stand-alone charges and currents, if the media had not modified them by its dielectric and/or magnetic polarization.

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[39] Similarly to the situation with the electric dipoles (see Eq. (3.24) and its discussion), it may be shown that the magnetic field of any closed current loop (or any system of such loops) satisfies the following equality:

$$\int_{r<R} \mathbf{B}(\mathbf{r}) d^3r = (2/3)\mu_0 \mathbf{m},$$

where the integral is over any sphere confining all the currents. On the other hand, for field (99), derived from the asymptotic approximation (90), such integral vanishes. In order to get a course-grain description of the magnetic field of a small system located at $r = 0$, which would be valid everywhere (though at $r \sim a$, only approximately), Eq. (99) should be modified as follows:

$$\mathbf{B}_{cg}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\mathbf{r}(\mathbf{r}\cdot\mathbf{m}) - \mathbf{m}r^2}{r^5} + \frac{8\pi}{3}\mathbf{m}\delta(\mathbf{r})\right).$$

Hence, strictly speaking, the macroscopic field **B** participating in Eq. (106) and beyond is the average *long-range* field of the magnetic dipoles (plus of the stand-alone currents **j**) rather than the genuine average magnetic field.

[40] This confusion is exacerbated by the fact that in Gaussian units, Eq. (108) has the form $\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$, and hence fields **B** and **H** has one dimensionality (and are equal in free space!) - though the unit of **H** has a different name (*oersted*, abbreviated as Oe). Mercifully, in the SI units, the dimensionality of **B** and **H** is different, with the unit of **H** being called *ampere per meter*.

Despite this similarity, let me note an important difference of signs in the relation (3.33) between **E**, **D**, and **P**, on one hand, and relation (108) between **B**, **H**, and **M**, on the other hand. It is *not* just the matter of definition. Indeed, due to the similarity of Eqs. (3.15), and (100), including similar signs, the electric and magnetic fields both try to orient the corresponding dipole moments along the field. Hence, in the media that allow such orientation (and as we will see momentarily, for magnetic media it is not always the case), the induced polarizations **P** and **M** are directed along, respectively, vectors **E** and **B**. According to Eq. (3.33), if the would-be field **D** is fixed - say, by a fixed stand-alone charge distribution $\rho(\mathbf{r})$ - such polarization *reduces* the genuine average electric field $\mathbf{E} = (\mathbf{D} - \mathbf{P})/\varepsilon_0$. On the other hand, Eq. (108) shows that in a magnetic media with fixed would-be field **H**, magnetic polarization with $\mathbf{M} \Uparrow \mathbf{B}$ *enhances* the average magnetic field $\mathbf{B} = (\mathbf{H} + \mathbf{M})/\mu_0$. This difference may be traced back to the sign difference in the initial relations (1.1) and (5.1), i.e. to the basic fact that charges of the same sign repulse, while currents of the same direction attract each other.

In order to form a complete system of differential equations, the macroscopic Maxwell equations (109) have to be complemented with "material relations" $\mathbf{D} \leftrightarrow \mathbf{E}$, $\mathbf{j} \leftrightarrow \mathbf{E}$, and $\mathbf{B} \leftrightarrow \mathbf{H}$. In previous two chapters we already discussed, in brief, two of them; let us proceed to the last one.

5.5. Magnetic materials

A major difference between the dielectric and magnetic material equations **D**(**E**) and **B**(**H**) is that while a typical dielectric media *reduces* the external electric field, magnetic media may *either reduce or enhance* it. In order to quantify this fact, let us consider the so-called *linear magnetics* in which **M** (and hence **H**) are proportional to **B**. Just as in dielectrics, in material without spontaneous magnetization, such linearity at relatively low fields follows from the Taylor expansion of function **M**(**B**). For isotropic materials, this proportionality is characterized by a scalar - either the *magnetic permeability* μ, defined by the following relation:

Magnetic permeability

$$\mathbf{B} \equiv \mu \mathbf{H}, \tag{5.110}$$

or the *magnetic susceptibility*[41] defined as

Magnetic susceptibility

$$\mathbf{M} = \chi_m \mathbf{H}. \tag{5.111}$$

Plugging these relations into Eq. (108), we see that these two parameters are not independent, but are related as

χ_m vs. μ

$$\mu = (1 + \chi_m)\mu_0. \tag{5.112}$$

Note that despite the superficial similarity between Eqs. (110)-(111) and relations (3.35)-(3.38) for linear dielectrics:

[41] According to Eq. (110) (i.e. in SI units), χ_m is dimensionless, while μ has the same the same dimensionality as μ_0. In the Gaussian units, μ is dimensionless, $(\mu)_{\text{Gaussian}} = (\mu)_{\text{SI}}/\mu_0$, and χ_m is also introduced differently, as $\mu = 1 + 4\pi\chi_m$, Hence, just as for the electric susceptibilities, these dimensionless coefficients are different in the two systems: $(\chi_m)_{\text{SI}} = 4\pi(\chi_m)_{\text{Gaussian}}$. Note also that χ_m is formally called the *volume* magnetic susceptibility, in order to distinguish it from the *molecular* susceptibility χ defined by a similar relation, $\mathbf{m} \equiv \chi\mathbf{H}$, where **m** is the average induced magnetic moment of a single dipole – e.g., a molecule. Evidently, in a dilute medium, i.e. in the absence of substantial dipole-dipole interaction, $\chi_m = n\chi$, where n is the dipole density.

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{P} = \chi_e \varepsilon_0 \mathbf{E}, \quad \varepsilon = (1 + \chi_e)\varepsilon_0, \tag{5.113}$$

there is an important conceptual difference between them. Namely, while vector **E** in the right-hand parts of Eqs. (113) is the real (average) electric field, vector **H** in the right-hand part of Eqs. (110)-(111) represents a "would-be" magnetic field, in all aspects similar to vector **D** rather than **E.** For relatively dense media, whose polarization may affect the genuine fields substantially, this difference between parameters ε and μ may make their properties (e.g., the Kramers-Kronig relations, to be discussed in Sec. 7.3) rather different.

Another difference between parameters ε and μ (and hence between χ_e and χ_m) is evident from Table 1 which lists the values of magnetic susceptibility for several materials. It shows that in contrast to linear dielectrics whose susceptibility χ_e is always positive, i.e. the dielectric constant $\varepsilon_r = \chi_e + 1$ is always larger than 1 (see Table 3.1), linear magnetics may be either *paramagnets* ($\chi_m > 0$, i. e. $\mu > \mu_0$) or *diamagnets* ($\chi_m < 0$, $\mu > \mu_0$).

Table 5.1. Magnetic susceptibility $(\chi_m)_{SI}$ of a few representative (and/or important) materials[(a)]

"Mu-metal" (75% Ni + 15% Fe + a few %% of Cu and Mo)	~20,000[(b)]
Permalloy (80% Ni + 20% Fe)	~8,000[(b)]
"Soft" (or "transformer") iron	~4,000[(b)]
Nickel	~100
Aluminum	$+2\times10^{-5}$
Diamond	-2×10^{-5}
Copper	-7×10^{-5}
Water	-9×10^{-6}
Bismuth (the strongest non-superconducting diamagnet)	-1.7×10^{-4}

[(a)]The table does not include bulk superconductors, which in a crude ("macroscopic") approximation may be described as perfect diamagnets (with **B** = 0, i.e. $\chi_m = -1$ and $\mu = 0$), though the actual physics of this phenomenon is more complex – see Sec. 6.3 below.

[(b)] The exact values of χ_m for soft ferromagnetic materials depend not only on their exact composition, but also on their thermal processing ("annealing"). Moreover, due to unintentional vibrations, the extremely high χ_m of such materials may somewhat decay with time, though may be restored to approach the original value by new annealing.

The reason of this difference is that in dielectrics, two different polarization mechanisms (schematically illustrated by Fig. 12) lead to the same sign of the average polarization. The first of them takes place in atoms without their own spontaneous polarization. A crude classical image of such an atom is an isotropic cloud of negatively charged electrons surrounding a positively charged nucleus - see Fig. 12a. The external electric field shifts the positive charge in the direction of **E**, and negative charges in the opposite direction, thus creating a dipole with aligned vectors **p** and **E**, and hence positive polarizability α_{mol} - see Eq. (3.39). As a result, the electric susceptibility is also positive – see Eqs. (3.41) or (3.71).

In the second case (Fig. 12b) of a gas or liquid consisting of *polar molecules*, each molecule has its own, spontaneous dipole moment $\mathbf{p}_0$ even in the absence of external electric field. (A typical example is a water molecule H_2O, with the positive oxygen ion positioned out of the line connecting two positive hydrogen atoms, thus producing a spontaneous dipole with moment's magnitude $p_0 \approx e\times 0.38\times 10^{-10}$m.) However, in the absence of the applied electric field, the orientation of such dipoles is random, so that the average polarization $\mathbf{P} = n\langle \mathbf{p}_0 \rangle$ equals zero. A weak applied field does not change the magnitude of the dipole moments significantly, but creates their preferential orientation along the field (in order to decrease the potential energy $U = -\mathbf{p}_0\cdot\mathbf{E}$), thus creating a nonvanishing vector average $\langle \mathbf{p}_0 \rangle$ directed along $\mathbf{E}$. If the applied field is not two high ($p_0E << k_BT$), the induced polarization $\mathbf{P} = n\langle \mathbf{p}_0 \rangle$ is proportional to $\mathbf{E}$, again giving a positive polarizability α_{mol}.[42]

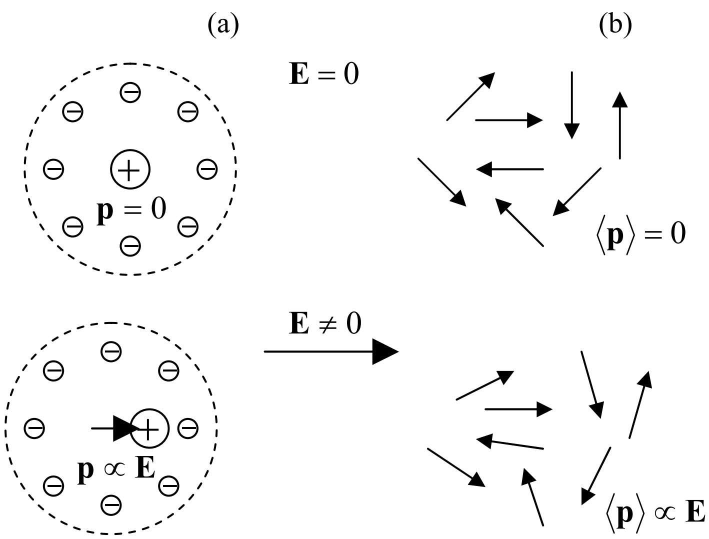

Fig. 5.12. Cartoons of two types of induced electrical polarization: (a) elementary dipole induction and (b) partial ordering of spontaneous elementary dipoles.

Returning to magnetics, the second of the above mechanisms, i.e. the ordering of spontaneous dipoles by the applied field, is responsible for the paramagnetism. Again, now according to Eq. (100), such field tends to align the dipoles along its direction, so that the average direction of spontaneous elementary moments $\mathbf{m}_0$, and hence the direction of $\mathbf{M}$, is the same as that of the average field $\mathbf{B}$ (i.e., for a diluted media, of $\mathbf{H} \approx \mathbf{B}/\mu_0$), resulting in a positive susceptibility χ_m. However, in contrast to the electric polarization, there is a mechanism of magnetic polarization, called the *orbital* (or "Larmor"[43]) *diamagnetism*, which gives $\chi_m < 0$. As its simplest model, let us consider the orbital motion of an atomic electron as classical particle of mass m_0, with electric charge q, about an immobile attractive center - modeling the atomic nucleus. As classical mechanics tells us, the central attractive force does

[42] The proportionality of $|\langle p_0 \rangle|$ (and hence P) to E is a result of a dynamic balance between the dipole-orienting torque (101) and disordering thermal fluctuations. A qualitative description of such balances is one of the main tasks of statistical mechanics - see, e.g., SM Chapters 2 and 4. However, the very fact of proportionality $P \propto E$ in low fields may be readily understood as the result of the Taylor expansion of function $P(E)$ at $E \to 0$.

[43] After J. Larmor (1857 – 1947) who first described the torque-induced precession mathematically.

not change particle's angular momentum $\mathbf{L} \equiv m_0\mathbf{r}\times\mathbf{v}$, but the applied magnetic field **B** (that may be taken uniform on the atomic scale) does, due to the torque (101) it applies to magnetic moment (95):

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = \mathbf{m}\times\mathbf{B} = \frac{q}{2m_0}\mathbf{L}\times\mathbf{B}\,. \tag{5.114}$$

The vector diagram in Fig. 13 shows that in the limit of relatively weak field, when the magnitude of the angular momentum **L** may be considered constant, this equation describes the rotation (called the *torque-induced precession*[44]) of vector **L** about the direction of vector **B**, with angular frequency $\boldsymbol{\Omega} = -q\mathbf{B}/2m_0$, independent on angle θ. Let me leave for the reader to use Eq. (114) for checking that, irrespectively the sign of charge q, the resulting additional magnetic moment $\Delta\mathbf{m}$ has a direction *opposite* to that of vector **B**, and hence χ_m is negative, leading to the Larmor diamagnetism.[45]

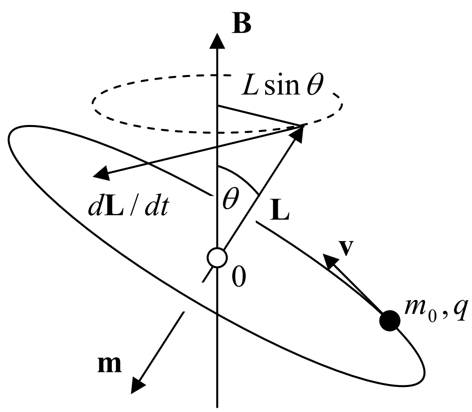

Fig. 5.13. Torque-induced precession of a charged particle in a magnetic field.

An important conceptual question is what exactly prevents the initial magnetic moment **m** that, according to Eq. (95), is associated with the angular momentum **L** of the electron, from turning along the magnetic field, just as in the second polarization mechanism illustrated by Fig. 12b - thus decreasing the potential energy (100) of the system. The answer is the same as for the usual mechanical top – it "wants" to fall due to the gravity field, but cannot do that due to the mechanical inertia. In classical physics, even a small friction (dissipation) eventually drains top's rotational kinetic energy, and it falls. However, in quantum mechanics the ground-state "motion" of electrons in an atom is not subjected to friction, because they cannot be brought to full rest due to Heisenberg's uncertainty principle. Somewhat counter-intuitively, the magnetic moments due to such fully-quantum effect as spin are much more susceptible to interaction with environment, so that in atoms with uncompensated spins, the magnetic dipole orientation mechanism prevails over the orbital diamagnetism, and the materials incorporating such atoms usually exhibit net paramagnetism – see Table 1.

Due to possible strong interactions between elementary dipoles, magnetism of materials is an extremely rich field of physics, with numerous interesting phenomena and elaborated theories.

[44] For a detailed discussion of the effect see, e.g., CM Sec. 6.5.

[45] The quantum-mechanical treatment (see, e.g., QM Sec. 6.4) confirms this qualitative picture, while giving quantitative corrections to the classical result for χ_m.

Unfortunately, all this physics is well outside the framework of this course, and I have to refer the interested reader to special literature,[46] but still need to mention its key notions.

Most importantly, a sufficiently strong dipole-dipole interaction may lead to their spontaneous ordering, even in the absence of the applied field. This ordering may correspond to either parallel alignment of the atomic dipoles (*ferromagnetism*) or anti-parallel alignment of the adjacent dipoles (*antiferromagnetism*). Evidently, the external effects of ferromagnetism are stronger, because such phase corresponds to a substantial spontaneous magnetization **M**. (This value is frequently called the *saturation magnetization*, $\mathbf{M}_s$, while the corresponding magnitude of $\mathbf{B} = \mu_0\mathbf{M}$ is called either the *saturation magnetic field*, or the *remanence field*, $\mathbf{B}_R$). The direction of $\mathbf{B}_R$ may switched by the application an external magnetic field, with a magnitude above certain value H_C called *coercivity*,[47] leading to the well-known hysteretic loops on the [B, H] plane - see Fig. 14 for a typical example.

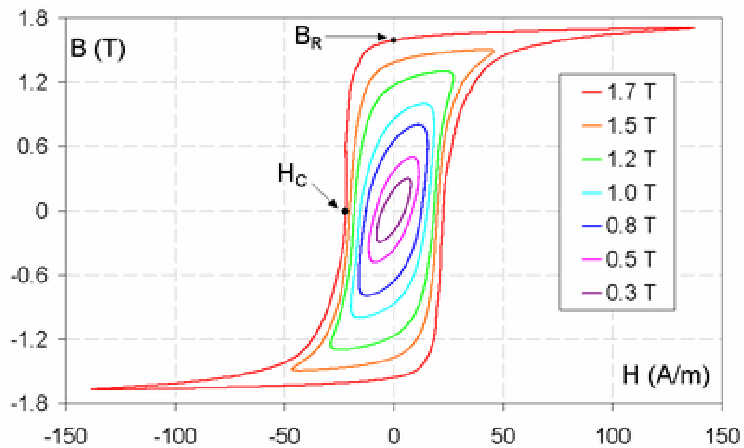

Fig. 5.14. Experimental magnetization curves of specially processed (cold-rolled) transformer steel, i.e. a solid solution of ~10% C and ~ 6% Si in Fe. (Adapted from www.thefullwiki.org/Hysteresis.)

In relatively low fields, $H << H_C$, such materials may be described as *hard (or* "permanent"*) ferromagnets*; at such approximate treatment, magnetization **M** is considered constant. On the other hand, the theory needed for a fair description phenomena at $H \sim H_C$ is rather complicated. Indeed, the direction of magnetization of crystals may be affected by the anisotropy of the crystal lattice. Because of that, typical non-crystalline ferromagnetic materials (like steel, permalloy, "mu-metal", etc.) consist of randomly oriented *magnetic domains*, each with certain spontaneous magnetization direction. The magnetic interaction of the domain with its neighbors and the external field determines the evolution of its magnetization and hence the average magnetic properties of the ferromagnet. In particular, such interaction explains why the hysteresis loop shape is dependent on the cycled field amplitude and cycling history – see Fig. 14. A very important class of multi-domain materials is the so-called *soft ferromagnets*, whose coercivity is relatively low. At low cycled field amplitude, the soft ferromagnets behave, on the average, as linear magnetics with very high values of χ_m and hence μ (see the top rows of Table 1, and Fig. 14) that are highly dependent on the material's fabrication technology and its post-fabrication thermal and mechanical treatments.

[46] See, e.g., D. J. Jiles, *Introduction to Magnetism and Magnetic Materials*, 2nd ed., CRC Press, 1998, or R. C. O'Handley, *Modern Magnetic Materials*, Wiley, 1999.

[47] Materials with very high coercivity H_C are frequently called *hard ferromagnets* or *permanent magnets*.

High values of χ_m are is also pertinent to magnetics in which the molecular dipole interaction is relatively weak, so that their ferromagnetic ordering may be destroyed by thermal fluctuations, if temperature is increased above the so-called *Curie temperature* T_C. At $T > T_C$, such materials behave as paramagnets, with susceptibility obeying the *Curie-Weiss law*

$$\chi_m \propto \frac{1}{T - T_C}. \qquad (5.115)$$

(At vanishing moment interaction, $T_C \to 0$, and Eq. (115) is reduced to the *Curie law* $\chi_m \propto 1/T$ typical for weak paramagnets.) The transition between the ferromagnetic and paramagnetic phase at $T = T_C$ is the classical example of *continuous phase transitions*, similar to that between the paraelectric and ferroelectric phases of a dielectric. In both cases, the "macroscopic" (average) polarization – either **M** or **P** – plays the role of the so-called *order parameter* that (in the absence of external fields) appears at $T = T_C$ and increases gradually at the further reduction of temperature.[48]

Before returning to magnetostatics per se, I have to mention the large practical role played by hard ferromagnetic materials (well beyond refrigerator magnets :-). Indeed, despite the decades of the exponential (*Moore's-law*) progress of semiconductor electronics, most computer data storage systems are still based on the *hard disk drives* whose active medium is a submicron-thin ferromagnetic layer, with bits stored in the form of the direction of the spontaneous magnetization of small film spots. This technology has reached a fantastic sophistication,[49] with recording data density approaching 10^{12} bits per square inch. Only recently it has started to be seriously challenged by the so-called *solid state drives* based on the flash semiconductor memories already mentioned in Chapter 3.

5.6. Systems with magnetics

Similarly to the electrostatics of linear dielectrics, magnetostatics of *linear* magnetics is very simple in the particular case when the stand-alone currents are deeply embedded into a medium with a constant permeability μ. Indeed, in this case, boundary conditions on the distant surface of the media do not affect the solution of the boundary problem described by the magnetic equations of the macroscopic Maxwell system (109). Now let us assume that we know the solution $\mathbf{B}_0(\mathbf{r})$ of the magnetic pair of the genuine ("microscopic") Maxwell equations (36) in free space, i.e. when the genuine current density **j** coincides with that of stand-alone currents. Then the macroscopic equations and the material equation (110) are completely satisfied with the pair of functions

$$\mathbf{H}(\mathbf{r}) = \frac{\mathbf{B}_0(\mathbf{r})}{\mu_0}, \quad \mathbf{B}(\mathbf{r}) = \mu\mathbf{H}(\mathbf{r}) = \frac{\mu}{\mu_0}\mathbf{B}_0(\mathbf{r}). \qquad (5.116)$$

Hence the only effect of a complete filling a system of fixed currents with a uniform, linear magnetic is the *increase* of the magnetic field **B** at all points by the same constant factor $\mu/\mu_0 \equiv 1 + \chi_m$. (As a reminder, a similar filling of a system of fixed charges with a uniform, linear dielectric leads to a *reduction* of the electric field **E** by factor $\varepsilon/\varepsilon_0 = \varepsilon_r = 1 + \chi_e$.)

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[48] A discussion of such transitions may be found, in particular, in SM Chapter 4.

[49] "*A magnetic head slider* [the read/write head – KKL] *flying over a [rather uneven – KKL] disk surface with a flying height of 25 nm with a relative speed of 20 meters/second is equivalent to an aircraft flying at a physical spacing of 0.2 µm at 900 kilometers/hour.*" B. Bhushan, as quoted in a (generally good) book by G. Hadjipanayis, *Magnetic Storage Systems Beyond 2000*, Springer, 2001.

However, this simple result is generally invalid in the case of non-uniform (or piece-wise uniform) magnetic samples. Theoretical analyses of magnetic field distribution in such non-uniform systems may be facilitated by two additional tools. First, integrating the macroscopic Maxwell equation (107) along a closed contour C limiting a smooth surface S, and using the Stokes theorem, we get the macroscopic version of the Ampère law (37):

Macroscopic Ampère law

$$\oint_C \mathbf{H} \cdot d\mathbf{r} = I. \tag{5.117}$$

This is exactly the replica of the "microscopic" equation Eq. (37), with the replacement $\mathbf{B}/\mu_0 \to \mathbf{H}$.

Let us apply this relation to a boundary between two regions with constant, but different μ, with no stand-alone currents on the border, similarly how this was done for field **E** in Sec. 3.4 - see Fig. 3.5. The result is similar as well:

$$H_\tau = \text{const}. \tag{5.118}$$

On the other hand, the integration of the Maxwell equation (29) over a Gaussian pillbox enclosing a border fragment (again similar to that shown in Fig. 3.5) yields the result similar to Eq. (3.46):

$$B_n = \text{const}, \quad \text{i.e. } \mu H_n = \text{const}. \tag{5.119}$$

Let us use these boundary conditions, first, to see what happens with a thin sheet of magnetic material (or any other strongly elongated sample) placed *parallel* to a uniform external field $\mathbf{H}_0$. Such sample cannot noticeably disturb the field in the free space outside it: $\mathbf{H}_{ext} = \mathbf{H}_0$, $\mathbf{B}_{ext} = \mathbf{H}_{ext}/\mu_0 = \mathbf{H}_0/\mu_0$. Now applying Eq. (118) to the dominating, large-area interfaces, we get $\mathbf{H}_{int} = \mathbf{H}_0$, i.e., $\mathbf{B}_{int} = (\mu/\mu_0)\, \mathbf{B}_0$.[50] The fact of constancy of field **H** in this geometry explains why this field is used as the horizontal axis in plots like Fig. 14: such measurements are typically carried out by placing an elongated sample of the material into the uniform field – say the one produced by a long solenoid.

Samples of other geometries may create strong perturbations of the external field, extended to distances of the order of the transversal dimensions of the sample. In order to analyze such problems, we may benefit from a simple, partial differential equation for a scalar function, e.g., the Laplace equation, because in Chapter 2 we have learned how to solve it for many simple geometries. In magnetostatics, the introduction of a scalar potential is generally impossible due to the vortex-like magnetic field lines, but if there are no stand-alone currents within the region we are interested in, then the Maxwell equation (32) for field **H** is reduced to $\nabla \times \mathbf{H} = 0$, and we may introduce the scalar potential of the magnetic field, ϕ_m, using the relation similar to Eq. (1.33):

$$\mathbf{H} = -\nabla \phi_m. \tag{5.120}$$

Combining it with the homogenous Maxwell equation for magnetic field, $\nabla \cdot \mathbf{B} = 0$, we arrive at the familiar differential equation,

$$\nabla \cdot (\mu \nabla \phi_m) = 0, \tag{5.121}$$

that, for a uniform media (μ = const), is reduced to our beloved Laplace equation. Moreover, Eqs. (118) and (119) give the very familiar boundary conditions: first

[50] The reader is highly encouraged to carry out a similar analysis of fields inside narrow gaps cut in a linear magnetic, similar to that carried out for linear dielectrics in Sec. 3.3 – see Fig. 3.6 and its discussion.

$$\frac{\partial \phi_m}{\partial \tau} = \text{const}, \tag{5.122a}$$

which is equivalent to

$$\phi_m = \text{const}, \tag{5.122b}$$

and also

$$\mu \frac{\partial \phi_m}{\partial n} = \text{const}. \tag{5.123}$$

Note that these boundary conditions are similar for (3.46) and (3.47) of electrostatics, with the replacement $\varepsilon \to \mu$.[51]

Let us analyze the geometric effects on magnetization, using the (too?) familiar structure: a sphere, made of a linear magnetic material, in a uniform external field. Since the differential equation and boundary conditions are similar to those of the similar electrostatics problem (see Fig. 3.8), we can use the above analogy to recycle the solution we already have got – see Eqs. (3.55)-(3.56). Just as in the electric case, the field outside the sphere, with potential

$$\left(\phi_m\right)_{r>R} = H_0\left(-r + \frac{\mu-\mu_0}{\mu+2\mu_0}\frac{R^3}{r^2}\right)\cos\theta, \tag{5.125a}$$

is a sum of the uniform external field $\mathbf{H}_0$ and the dipole field (99) with the following induced magnetic dipole moment of the sphere:[52]

$$\mathbf{m} = 4\pi \frac{\mu-\mu_0}{\mu+2\mu_0} R^3 \mathbf{H}_0. \tag{5.125b}$$

On the contrary, the internal field is perfectly uniform:

$$\left(\phi_m\right)_{r<R} = -H_0 \frac{3\mu_0}{\mu+2\mu_0} r\cos\theta, \quad \frac{H_{\text{int}}}{H_0} = \frac{3\mu_0}{\mu+2\mu_0}, \qquad \frac{B_{\text{int}}}{B_0} = \frac{\mu H_{\text{int}}}{\mu_0 H_0} = \frac{3\mu}{\mu+2\mu_0}. \tag{5.126}$$

Note that though $\mathbf{H}$ inside the sphere is not equal to its value of the external field $\mathbf{H}_0$. This example shows that the interpretation of $\mathbf{H}$ as the "would-be" magnetic field generated by external

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[51] This similarity may seem strange, because earlier we have seen that parameter μ is physically more similar to $1/\varepsilon$. The reason for this paradox is that in magnetostatics, the introduced potential ϕ_m is traditionally used to describe the "would-be field" $\mathbf{H}$, while in electrostatics, potential ϕ describes the real (average) electric field $\mathbf{E}$. (This tradition persists from the old days when $\mathbf{H}$ was perceived as a genuine magnetic field.)

[52] Instead of differentiating the ϕ_m given by Eq. (125a), we may use the absolute similarity of Eqs. (3.13) and (99), to derive from Eq. (3.17) a similar expression for the magnetic potential of an arbitrary magnetic dipole:

$$\phi_m = \frac{1}{4\pi}\frac{m\cos\theta}{r^2}.$$

Now comparing this formula with the second term of Eq. (125a), we immediately get Eq. (125b).

currents **j** should not be exaggerated into saying that its distribution is independent on the magnetic bodies in the system.[53]

In the limit $\mu >> \mu_0$, Eqs. (126) yield $H_{int}/H_0 << 1$, $B_{int}/H_0 = 3\mu_0$, the factor 3 being specific for the particular geometry of the sphere. If a sample is stretched along the applied field, this limitation of the field concentration is gradually removed, and B_{int} tends to its maximum value $\mu H_0 >> B_{ext}$, as was discussed above. This effect of "magnetic line concentration" in high-μ materials is used in such practically important devices as transformers, in which two multi-turn coils are wound on a ring-shaped (e.g., toroidal, see Fig. 6b) core made of a soft ferromagnetic material (such as the *transformer steel*, see Table 1) with $\mu >> \mu_0$. This minimizes the number of "stray" field lines, and makes the magnetic flux Φ piercing each wire turn (of either coil) virtually the same – the equality important for secondary voltage induction – see the next chapter.

The second theoretical tool, frequently useful for problem solution, is a macroscopic expression for magnetic field energy U. For a system with linear magnetic materials, we may repeat the transformation of Eq. (55), made in Sec. 3, but with due respect to the magnetization, i.e. replacing **j** not from Eq. (56), but from Eq. (107). As a result, instead of Eq. (57) we get

Field energy in a linear magnetic

$$U = \int_V u(\mathbf{r}) d^3 r, \quad \text{with } u = \frac{\mathbf{B}\cdot\mathbf{H}}{2} = \frac{B^2}{2\mu} = \frac{\mu H^2}{2}, \tag{5.127}$$

This result is evidently similar to Eq. (3.79) of electrostatics.

For the general case of nonlinear magnetics, calculations similar to those resulting in Eq. (3.82) give the following analog of that relation:

General magnetic energy variation

$$\delta u = \mathbf{H}\cdot\delta\mathbf{B}, \tag{5.128}$$

for a linear magnetic yielding Eq. (127). Similarly to the electrostatics of dielectrics, we may argue that according to Eq. (128), in systems with magnetic media, **H** plays the role of the generalized force, and **B** of the generalized coordinate (per unit volume).[54] As the result, the Gibbs potential energy, whose minimum corresponds to the stable equilibrium of the system in an external field $\mathbf{H}_{ext}$, is

Gibbs potential energy

$$\mathscr{G} = \int_V g(\mathbf{r}) d^3 r, \quad \text{with } g(\mathbf{r}) \equiv u(\mathbf{r}) - \mathbf{H}_{ext}\cdot\mathbf{B}, \tag{5.129}$$

the expression to be compared with Eq. (3.84). Similarly, for a system with linear magnetics, the latter of these expressions may be integrated over the variations to give

[53] From the standpoint of mathematics, this happens because the solution to a boundary problem is determined by not only the differential equation inside the system (in our case, the Laplace equation for potential ϕ_m), but also by boundary conditions – which are affected by magnetics – see Eqs. (118)-(119).

[54] Note that in this respect, the analogy with electrostatics is incomplete. Indeed, according to Eq. (3.82), in electrostatics the role of a generalized coordinate is played by would-be field **D**, and that of the generalized force, by the real (average) electric field **E**. This difference may be traced back to the fact that electric field **E** may perform work on a moving charged particle, while the magnetic part of the Lorentz force (10), $\mathbf{v}\times\mathbf{B}$, is always perpendicular to particle's velocity, and its work equals zero. However, this difference does not affect the full analogy of expressions (3.79) and (127) for field energy density in linear media.

$$g(\mathbf{r}) = \frac{1}{2\mu}\mathbf{B}\cdot\mathbf{B} - \mathbf{H}_{\text{ext}}\cdot\mathbf{B} = \frac{1}{2\mu}(\mathbf{B} - \mu\mathbf{H}_{\text{ext}})^2 + \text{const}, \tag{5.130}$$

with similar consequences for the external magnetic field penetration into a system with magnetics. As a sanity check, for a uniform system with negligible fringe fields, such as a long solenoid filled with a uniform, linear magnetic material, Eq. (130) may be readily integrated over the sample volume to give

$$\mathscr{G}(\mathbf{r}) = \frac{1}{2\mu}(\mathbf{B} - \mu\mathbf{H}_{\text{ext}})^2 V + \text{const}, \tag{5.131}$$

so that the minimum of the Gibbs potential energy, i.e. the stable equilibrium of the system, corresponds to the result that has already been derived in the beginning of this section: $\mathbf{B} = \mu\mathbf{H}_{\text{ext}}$, i.e. $\mathbf{H} = \mathbf{H}_{\text{ext}}$.

For the important particular case of a long solenoid (Fig. 6a) filled with a linear magnetic material, we hay find field H from Eq. (117), just as we used Eq. (37) in Sec. 2 for finding B for a similar empty solenoid, getting

$$H = In, \text{ and hence } B = \mu In. \tag{5.132}$$

Now we may plug this result into Eq. (127) to calculate the magnetic energy stored in the solenoid:

$$U = uV = \frac{\mu H^2}{2} lA = \frac{\mu(nI)^2 lA}{2}, \tag{5.132}$$

and then use Eq. (72) to calculate its self-inductance:

$$L = \frac{U}{I^2/2} = \mu n^2 lA \tag{5.133}$$

- as evident generalization of Eq. (75). This result explains why filling of solenoids with soft ferromagnets with $\mu >> \mu_0$ is so popular in the electrical engineering practice, where large self- and mutual inductances are frequently needed in systems with size and/or weight restrictions.

Now, let us use these two tools to discuss a curious (and practically important) approach to systems with ferromagnetic cores. First, let us find the magnetic flux Φ in a system with a relatively thin, closed magnetic core made of sections of (possibly, different) soft ferromagnets, with the cross-section areas A_k much smaller than the squared lengths l_k of the sections - see Fig. 15.

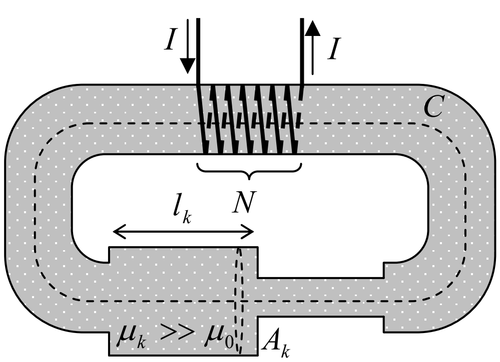


Fig. 5.15. Deriving the “magnetic Ohm law” (135).

If all $\mu_k >> \mu_0$, virtually all field lines are confined to the interior of the core. Then, applying the macroscopic Ampère law (117) to contour C, which follows a magnetic field line inside the core (see the dashed line in Fig. 15), we get the following approximate expression (exactly valid only in the limit μ_k/μ_0, $l_k^2/A_k \to \infty$):

$$\oint_C H_l dl \approx \sum_k l_k H_k = \sum_k l_k \frac{B_k}{\mu_k} = NI . \tag{5.134}$$

However, since the magnetic field lines stay in the core, the magnetic flux $\Phi_k \approx B_k A_k$ should be the same ($\equiv \Phi$) for each section, so that $B_k = \Phi/A_k$. Plugging this condition into Eq. (134), we get

Magnetic Ohm law and reluctance

$$\Phi = \frac{NI}{\sum_k \mathcal{R}_k}, \quad \text{where } \mathcal{R}_k \equiv \frac{l_k}{\mu_k A_k} . \tag{5.135}$$

Note a close analogy of the first of these equations with the Ohm law for several resistors connected in series, with the magnetic flux playing the role of electric current, while the product NI, of the voltage applied to the resistor chain. This analogy is fortified by the fact that the second of Eqs. (135) is similar to the expression for resistance $R = l/\sigma A$ of a long uniform conductor, with the magnetic permeability μ playing the role of the electric conductivity σ. (In order to sound similar, but still different from resistance R, parameter $\mathcal{R}$ is called the *reluctance*.) This is why Eq. (135) is called the *magnetic Ohm law*; it is very useful for approximate analyses of systems like ac transformers, magnetic energy storage systems, etc.

The role of the "magnetic e.m.f." NI may be also played by a permanent-magnet section of the core. Indeed, for relatively low fields we may use the Taylor expansion of the nonlinear function $B(H)$ near $H = 0$ to write

$$B \approx \mp \mu_0 M_s + \mu_d H, \quad \mu_d \equiv \frac{dB}{dH}\Big|_{H=0} , \tag{5.136}$$

where M_s is the spontaneous magnetization magnitude at $H = 0$, the $\mp$ sign corresponds to two possible directions of the magnetization, and parameter μ_d is called the *differential* (or "dynamic") permeability. Expressing H from this relation, and using it in one of components of the sum (134), we again get a result similar to Eq. (135)

$$\Phi = \mp \frac{(NI)_{\text{ef}}}{\mathcal{R}_H + \sum_k \mathcal{R}_k}, \quad \text{with } \mathcal{R}_H \equiv \frac{l_H}{A_H \mu_d}, \tag{5.137}$$

where l_H and A_H are geometric dimensions of the hard-ferromagnet section, and product NI is replaced with its effective value

$$(NI)_{\text{ef}} = \mp \frac{\mu_0}{\mu_d} M_s l_H . \tag{5.138}$$

This result may be used for a semi-quantitative explanation of the well-known short-range forces acting between permanent magnets (or between them and soft ferromagnets) at their mechanical contact (Fig. 16).

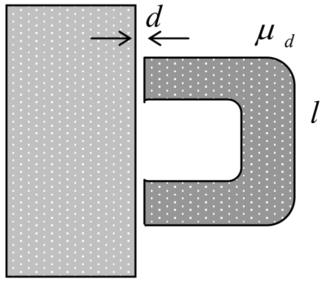


Fig. 5.16. Short-range interaction between magnets.

Indeed, considering the free-space gaps between them as sections of the core (which is approximately correct, because due to the small gap thickness d the magnetic field lines cannot stray far from the contact area), and neglecting the reluctance $\mathcal{R}$ of the bulk material (due to its larger cross-section), we get

$$\left|\Phi\right| \propto \left(\frac{2d}{\mu_0}+\frac{l}{\mu_d}\right)^{-1}, \tag{5.139}$$

so that, according to Eq. (127), the magnetic energy of the system (disregarding the constant energy of the permanent magnetization) is

$$U \propto \left(\frac{2d}{\mu_0}+\frac{l}{\mu_d}\right)B^2 \propto \left(\frac{2d}{\mu_0}+\frac{l}{\mu_d}\right)^{-1} \propto \frac{1}{d+d_0}, \quad d_0 \equiv \frac{1}{2}\frac{\mu_0}{\mu_d}l << l\,. \tag{5.140}$$

Hence the magnet attraction force,

$$F = \left|\frac{\partial U}{\partial d}\right| \propto \frac{1}{(d+d_0)^2}, \tag{5.141}$$

behaves almost as the divergence $1/d^2$ truncated at a short distance $d_0 << l$. Due to that truncation, the force is finite at $d = 0$; this exactly the force you need to apply to detach two magnets.

Finally, let us discuss in brief a related effect in experiments with thin and long hard ferromagnetic samples - "needles", like those used in magnetic compasses. Using the definition (108) of field **H**, the Maxwell equation (29) takes the form

$$\nabla\cdot\mathbf{B} \equiv \mu_0\nabla\cdot(\mathbf{H}+\mathbf{M}) = 0, \tag{5.142}$$

and may be rewritten as

$$\nabla\cdot\mathbf{H} = -\nabla\cdot\mathbf{M}\,. \tag{5.143}$$

While this relation is general, it is especially convenient in hard ferromagnets, where **M** is virtually fixed by the saturation. Comparing this equation with Eq. (1.27) for the electrostatic field, we see that the right-hand part of Eq. (143) may be considered as a fixed source of a Coulomb-like magnetic field.

For example, let us apply Eq. (143) to a thin, long needle made of a hard ferromagnet (Fig. 17a). Inside the needle, $\mathbf{M} = \mathbf{M}_s$ = const, while outside it $\mathbf{M} = 0$, so that the right-hand part of Eq. (143) is substantially different from zero only in two small areas at the needle's ends, and on much larger distances we can use the following approximation:

$$\nabla\cdot\mathbf{H} = -q_m\delta(\mathbf{r}-\mathbf{r}_1) + q_m\delta(\mathbf{r}-\mathbf{r}_2), \tag{5.155}$$

where $\mathbf{r}_{1,2}$ are ends' positions, and $q_m \equiv M_s A$, with A being the needle's cross-section area. This equation is completely similar to Eq. (1.27) for the electric field created by two equal and opposite point charges. In particular, if two ends of two needles are hold at an intermediate distance r ($A^{1/2} << r << l$, where l is the needle length, see Fig. 17b), the ends interact in accordance with the *magnetic Coulomb law*

$$F \propto \frac{q_m^2}{r^2} = \frac{M_s^2 A^2}{r^2}. \qquad (5.156)$$

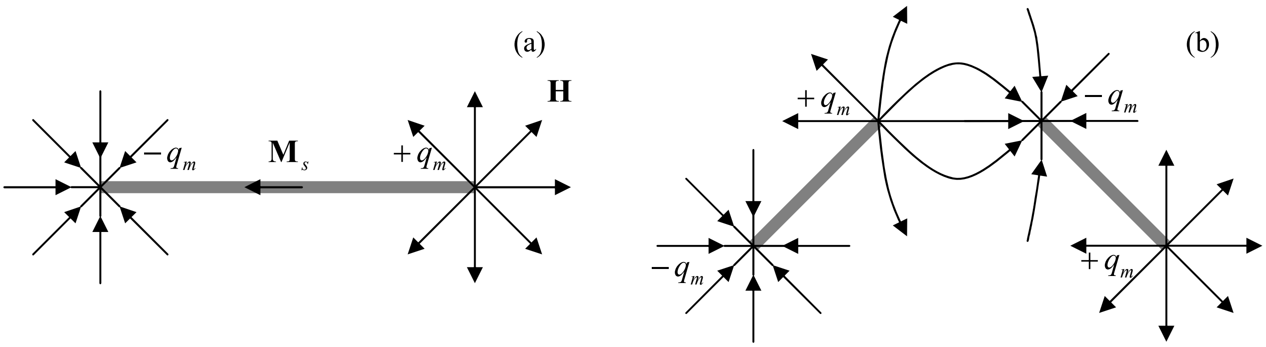
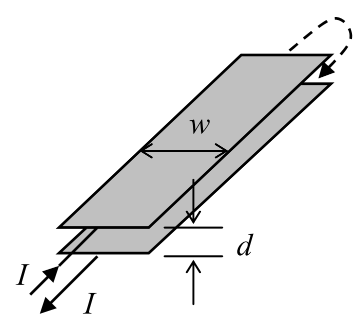


Fig. 5.17. (a) "Magnetic charges" at the ends of a thin ferromagnetic needle and (b) the result of its breaking into two parts (schematically).

The "only" (but conceptually, very significant!) difference with electrostatics is that the "magnetic charges" $\pm q_m$ cannot be fully separated. For example, if we break a magnetic needle in the middle in at attempt to bring its two ends further apart, two new "charges" appear – see Fig. 17b. There are several solid state systems where more flexible structures, similar to the magnetic needles, may be implemented. First of all, certain ("type-II") superconductors may sustain so-called *Abrikosov vortices* – crudely, flexible tubes with field-suppressed superconductivity inside, each carrying one magnetic flux quantum $\Phi_0 = \hbar/\pi e \approx 2\times10^{-15}$ Wb – see Sec. 6.3. Ending on superconductor's surface, these tubes let the magnetic field lines to spread into the surrounding space, essentially forming a magnetic monopole analog (of course, with an equal and opposite "monopole" on another end of the line). Such flux tubes are not only flexible but readily stretchable, resulting in several peculiar effects.[55] Another, recently found, examples of paired "monopoles" include *spin chains* in so-called *spin ices* – crystals with paramagnetic ions arranged into a specific (pyrochlore) lattice – such as dysprosium titanate $Dy_2Ti_2O_7$.[56]

5.7. Exercise problems

5.1. Two straight, parallel, long, plane, thin strips of width d, separated by distance d, are used to form a current loop - see Fig. on the right. Calculate the magnetic field in the plane located at the middle between the planes of the strips, assuming that current I is uniformly distributed across strip width.

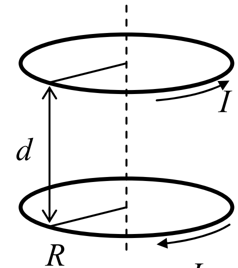


[55] A detailed discussion of the Abrikosov vortices may be found, for example, in Chapter 5 of M. Tinkham, *Introduction to Superconductivity*, 2nd ed., McGraw-Hill, 1996.
[56] See, e.g., L. Jaubert and P. Holdworth, *J. Phys. – Cond. Matt.* **23**, 164222 (2011) and references therein.

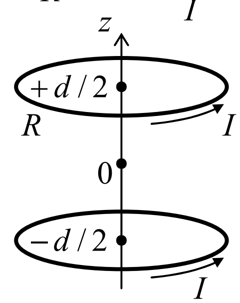
5.2. For the system studied in the previous problem, but now only in the limit $d << w$, calculate:

(i) the distribution of the magnetic field (in the simplest possible way),
(ii) the vector-potential of the field,
(iii) the force (per unit length) acting on each strip, and
(iv) the magnetic energy and self-inductance of the system (per unit length).

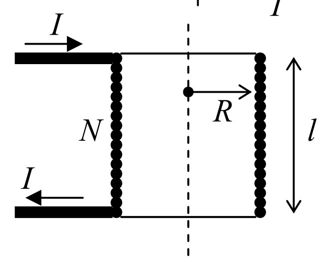
5.3. Calculate the magnetic field distribution near the center of the system of two similar, plane, round, coaxial wire coils, fed by equal but oppositely directed currents – see Fig. on the right.



5.4. The two-coil-system, similar to that considered in the previous problem, carries equal and similarly directed currents – see Fig. on the right. Calculate what should be the ratio d/R for the second derivative $\partial^2 B_z/\partial z^2$ at $z = 0$ to vanish.[57]



5.5. Calculate the magnetic field distribution along the axis of a straight solenoid (see Fig. 6a, partly reproduced on the right) with a finite length l, and round cross-section of radius R. Assume that the solenoid has many wire turns ($N >> 1$) that are uniformly distributed along its length.

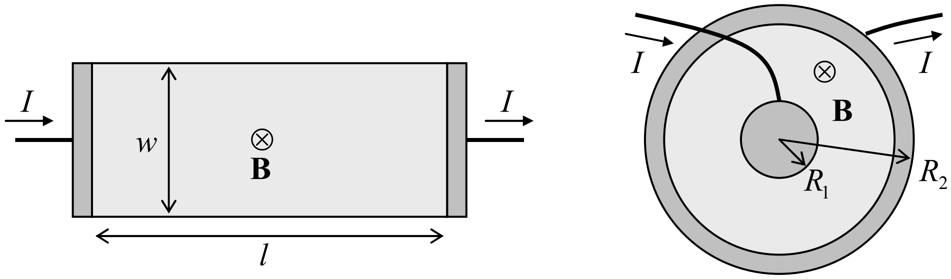


5.6. A thin spherical shell of radius R, with charge Q uniformly distributed over its surface, rotates about its axis with angular velocity ω. Calculate the distribution of the magnetic field everywhere in space.

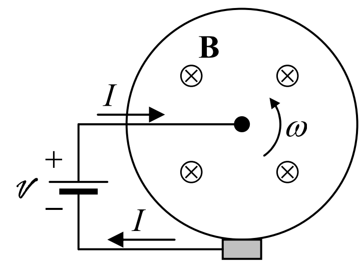
5.7. A sphere of radius R, made of an insulating material with a uniform electric charge density ρ, rotates about its diameter with angular velocity ω. Calculate the magnetic field distribution inside the sphere and outside it.

5.8. The reader is (hopefully :-) familiar with the classical Hall effect when it takes place in the usual rectangular *Hall bar* geometry – see the left panel of the Fig. below. However, the effect takes a different form in the so-called *Corbino disk* – see the right panel below. (Dark shading shows electrodes, with no appreciable resistance.) Analyze the effect in both geometries, assuming that in both cases the conductors are thin, planar, have a constant Ohmic conductivity σ and charge carrier density n, and that the applied magnetic field **B** is uniform and normal to conductors' planes.

[57] Such system, producing a highly uniform field near its center, is called the *Helmholtz coils*, and is broadly used in physics experiment.



5.9.* The simplest model of the famous *homopolar motor*[58] is a thin, round conducting disk, placed into a uniform magnetic field normal to its plane, and fed by dc current flowing from disk's center to a sliding electrode ("brush") – see Fig. on the right.



(i) Express the torque, rotating the disk, via its radius $\mathcal{R}$, magnetic field **B**, and current *I*.

(ii) If the disk is allowed to rotate about its axis, and the motor is driven by a battery with e.m.f. $\mathcal{V}$, calculate its angular velocity ω, neglecting electric circuit's resistance and friction.

(iii) Now assuming that the current circuit (battery + wires + contacts + disk itself) has full resistance R, derive and solve the equation for the time evolution of ω, and analyze the solution.

5.10.* Estimate the values of magnetic susceptibility due to

(i) orbital diamagnetism, and
(ii) spin paramagnetism,

for a dilute medium with negligible interaction between molecular dipoles.

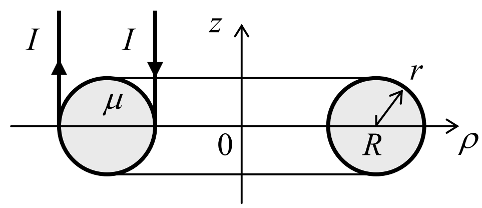
Hints: For task (i), you may use the classical model described by Eq. (114) (see Fig. 13), while for task (ii), assume the mechanism of ordering of spontaneous magnetic dipoles $\mathbf{m}_0$, similar to the one sketched for electric dipoles in Fig. 12b, with the magnitude of the order of the Bohr magneton μ_B – see Eq. (96).

5.11.* Use the classical picture of the orbital ("Larmor") diamagnetism, discussed in Sec. 5.5 of the lecture notes, to calculate its (small) correction $\Delta\mathbf{B}(0)$ to the magnetic field **B**, as felt by the atomic nucleus, modeling atomic electrons by a spherically-symmetric cloud with electric charge density $\rho(r)$. Express the result via the value $\phi(0)$ of the electrostatic potential of electrons' cloud, and use this expression for a crude numerical estimate of the relative correction, $\Delta B(0)/B$, for the hydrogen atom.

5.12. Current *I* is flows in a thin wire bent into a plane, round loop of radius *R*. Calculate the net magnetic flux through the whole plane in which the loop is located.

[58] It was invented by M. Faraday in 1821, i.e. well before his celebrated work on electromagnetic induction. The adjective "homopolar" refers to the constant "polarity" (sign) of the current; the alternative term is "unipolar".

5.13. Calculate the (self-) inductance of a toroidal solenoid (Fig. 6b) with the round cross-section of radius $r \sim R$ (see Fig. on the right), filled with a material of magnetic permeability μ, with many ($N >> 1$, R/r) wire turns uniformly distributed along the perimeter. Check your results by analyzing the limit $r << R$.



Hint : You may like to use the following table integral:[59]

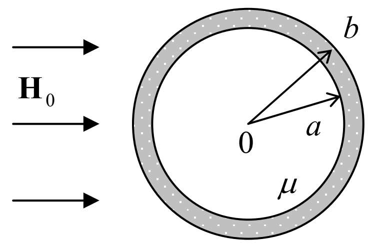
$$\int_0^1 \ln \frac{a+\left(1-\xi^2\right)^{1/2}}{a-\left(1-\xi^2\right)^{1/2}} d\xi = \pi\left[a-\left(a^2-1\right)^{1/2}\right], \quad \text{for } a \geq 1.$$

5.14. Prove that:

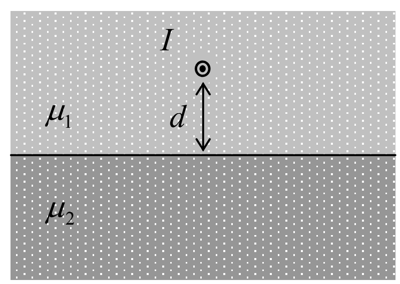
(i) the self-inductance L of a current loop cannot be negative, and

(ii) each mutual inductance coefficient $L_{kk'}$, defined by Eq. (60), cannot be larger than $(L_{kk}L_{k'k'})^{1/2}$.

5.15. A round cylindrical shell, made of a soft ferromagnet, is placed into a uniform external field $\mathbf{H}_0$ perpendicular to its axis - see Fig. on the right. Find the distribution of the magnetic field everywhere in the system, and discuss its efficiency as a "magnetic shield".



5.16. A straight thin wire, carrying current I, passes parallel to the plane boundary between two uniform, linear magnetics – see Fig. on the right. Calculate the magnetic field everywhere in the system, and the force (per unit length) exerted on the wire.



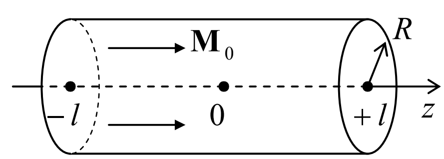
5.17. Calculate the distribution of magnetic field around a sphere made of a hard ferromagnet with a permanent, uniform magnetization $\mathbf{M}$ = const.

5.18.* A limited volume V is filled with a magnetic material with magnetization $\mathbf{M}(\mathbf{r})$.

(i) Use Eq. (5.143) to write explicit expressions for the magnetic field and its potential, induced by the magnetization.

(ii) Recast these expressions in forms convenient when $\mathbf{M}(\mathbf{r}) = \mathbf{M}_0$ = const inside volume V.

5.19. Use the results of the previous problem to calculate the distribution of the magnetic field along the axis of a straight



[59] See, e.g., MA (6.13).

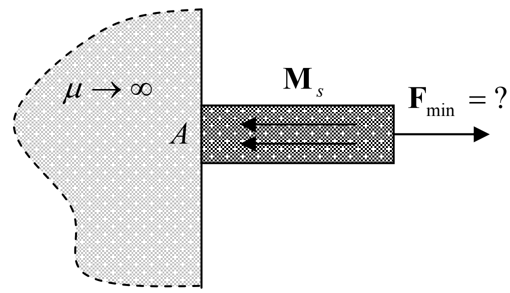
permanent magnet of length $2l$, with round cross-section of radius R, and uniform magnetization $\mathbf{M}_0$ parallel to the axis - see Fig. on the right.

5.20. A very broad film of thickness $2t$ is magnetized normally to its plane, with a periodic checkerboard pattern with square side a:

$$\mathbf{M}\Big|_{|z|<t} = \mathbf{n}_z M(x,y), \qquad \text{with } M(x,y) = M_0 \times \begin{cases} (+1), & \text{if } \cos\dfrac{\pi x}{a}\cos\dfrac{\pi y}{a} > 0, \\ (-1), & \text{if } \cos\dfrac{\pi x}{a}\cos\dfrac{\pi y}{a} < 0. \end{cases}$$

Calculate the magnetic field distribution in space.[60]

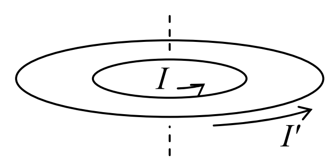
5.21. A flat end of a rod magnet, with cross-section area A, with saturated magnetization $\boldsymbol{M}_s$ directed along rod's length, is let to stuck to a plane surface of a large sample made of a soft ferromagnetic material with $\mu >> \mu_0$. Calculate the force necessary to detach the rod from the surface, if it is applied strictly perpendicular to the contact surface – see Fig. on the right.



5.22.* Based on the discussion of the quadrupole electrostatic lens in Sec. 2.4 of the lecture notes, suggest permanent-magnet systems which may similarly focus particles moving close to system's axis, and carrying:

(i) an electric charge,
(ii) no net electric charge, but a nonvanishing spontaneous magnetic dipole moment $\mathbf{m}$.

5.23. A circular wire loop, carrying a fixed dc current, has been placed inside a similar but larger loop, carrying a fixed current in the same direction – see Fig. on the right. Use semi-quantitative arguments to analyze the mechanical stability of the coaxial, coplanar position of the inner loop with respect to its possible angular, axial, and lateral displacements, if the position of the outer loop is fixed.



[60] This problem is of an evident relevance for the *perpendicular magnetic recording* (PMR) technology, which presently dominates the high-density digital magnetic recording, with the density already approaching 1 Tb/in^2.

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