Chapter 3. A Few Simple Problems

In this chapter, I will review the solutions of a few simple but very important problems of particle motion, that may be reduced to one dimension, including the famous "planetary" problem of two particles interacting via a spherically-symmetric potential. In the process, we will discuss several methods that will be useful for the analysis of more complex systems.

3.1. One-dimensional and 1D-reducuble systems

If a particle is confined to motion along a straight line (say, axis x), its position, of course, is completely defined by this coordinate. In this case, as we already know, particle's Lagrangian is given by Eq. (2.21):

$$L = T(\dot{x}) - U(x,t), \quad T(\dot{x}) = \frac{m}{2}\dot{x}^2, \tag{3.1}$$

so that the Lagrange equation of motion (2.22)

$$m\ddot{x} = -\frac{\partial U(x,t)}{\partial x} \tag{3.2}$$

is just the x-component of the 2nd Newton law.

It is convenient to discuss the dynamics of such *really 1D* systems in the same breath with that of *effectively 1D systems* whose position, due to holonomic constraints and/or conservation laws, is also fully determined by one generalized coordinate q, and whose Lagrangians may be presented in a form similar to Eq. (1):

Effectively-1D system

$$L = T_{\text{ef}}(\dot{q}) - U_{\text{ef}}(q,t), \quad T_{\text{ef}} = \frac{m_{\text{ef}}}{2}\dot{q}^2, \tag{3.3}$$

where m_{ef} is some constant which may be considered as the *effective mass* of the system, and the function U_{ef} its *effective potential energy*. In this case the Lagrange equation (2.19) describing the system dynamics has a form similar to Eq. (2):

$$m_{\text{ef}}\ddot{q} = -\frac{\partial U_{\text{ef}}(q,t)}{\partial q}. \tag{3.4}$$

As an example, let us return again to our testbed system shown in Fig. 1.5. We have already seen that for that system, having one degree of freedom, the genuine kinetic energy T, expressed by the first of Eqs. (2.23), is *not* a quadratically-homogeneous function of the generalized velocity. However, the system's Lagrangian (2.23) still may be presented in form (3),

$$L = \frac{m}{2}R^2\dot{\theta}^2 + \frac{m}{2}R^2\omega^2\sin^2\theta + mgR\cos\theta + \text{const} = T_{\text{ef}} - U_{\text{ef}}, \tag{3.5}$$

if we take

$$T_{\text{ef}} \equiv \frac{m}{2}R^2\dot{\theta}^2, \quad U_{\text{ef}} \equiv -\frac{m}{2}R^2\omega^2\sin^2\theta - mgR\cos\theta + \text{const}. \tag{3.6}$$

In this new partitioning of function L, which is legitimate because U_{ef} depends only on the generalized coordinate θ, but not on the corresponding generalized velocity, T_{ef} includes only a part of the full kinetic energy T of the bead, while U_{ef} includes not only the real potential energy U of the bead in the gravity field, but also an additional term related to ring rotation. (As we will see in Sec. 6.6, this term may be interpreted as the effective potential energy due to the inertial centrifugal "force".)

Returning to the general case of effectively 1D systems with Lagrangian (3), let us calculate their Hamiltonian function, using its definition (2.32):

$$H = \frac{\partial L}{\partial \dot{q}}\dot{q} - L = m_{\text{ef}}\dot{q}^2 - (T_{\text{ef}} - U_{\text{ef}}) = T_{\text{ef}} + U_{\text{ef}}. \qquad (3.7)$$

So, H is expressed via T_{ef} and U_{ef} exactly as the mechanical energy E is expressed via genuine T and U.

3.2. Equilibrium and stability

Autonomous systems are defined as the dynamic systems whose equations of motion do not depend on time. For 1D (and effectively 1D) systems obeying Eq. (4), this means that their function U_{ef}, and hence the Lagrangian function (5) should not depend on time explicitly. According to Eqs. (2.35), in such systems the Hamiltonian function (7), i.e. the sum $T_{\text{ef}} + U_{\text{ef}}$, is an integral of motion. However, be careful! This may not be true for system's mechanical energy E; for example, as we already know from Sec. 2.2, for our testbed problem, with the generalized coordinate $q = \theta$ (Fig. 2.1), $H \neq E$.

According to Eq. (4), an autonomous system, at appropriate initial conditions, may stay in equilibrium at one or several *stationary* (alternatively called *fixed*) *points* q_n, corresponding to either the minimum or a maximum of the effective potential energy (see Fig. 1):

$$\frac{dU_{\text{ef}}}{dq}(q_n) = 0. \qquad (3.8)$$

Fixed-point condition

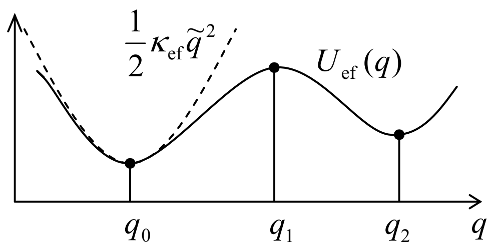


Fig. 3.1. Effective potential energy profile near stable (q_0, q_2) and unstable (q_1) fixed points, and its quadratic approximation (10) near point q_0 – schematically.

In order to explore the *stability* of such fixed points, let us analyze the dynamics of small deviations

$$\tilde{q}(t) \equiv q(t) - q_n \qquad (3.9)$$

from the equilibrium. For that, let us expand function $U_{\text{ef}}(q)$ in the Taylor series at a fixed point,

$$U_{\text{ef}}(q) = U_{\text{ef}}(q_n) + \frac{dU_{\text{ef}}}{dq}(q_n)\tilde{q} + \frac{1}{2}\frac{d^2U_{\text{ef}}}{dq^2}(q_n)\tilde{q}^2 + \dots. \qquad (3.10)$$

The first term in the right-hand part, $U_{ef}(q_n)$, is arbitrary and does not affect motion. The next term, linear in deviation $\tilde{q}$, is equal zero – see the fixed point definition (8). Hence the fixed point stability is determined by the next term, quadratic in $\tilde{q}$, more exactly by its coefficient,

$$\kappa_{ef} \equiv \frac{d^2 U_{ef}}{dq^2}(q_n) \tag{3.11}$$

which plays the role of the effective spring constant. Indeed, neglecting the higher terms of the Taylor expansion (10),[1] we see that Eq. (4) takes the familiar form - cf. Eq. (1.1):

$$m_{ef}\ddot{\tilde{q}} + \kappa_{ef}\tilde{q} = 0. \tag{3.12}$$

I am confident that the reader of these notes knows everything about this equation, but since we will soon run into similar but more complex equations, let us review the formal procedure of its solution. From the mathematical standpoint, Eq. (12) is an ordinary, linear differential equation of the second order, with constant coefficients. The theory of such equations tells us that its general solution (for any initial conditions) may be presented as

$$\tilde{q}(t) = c_+ e^{\lambda_+ t} + c_- e^{\lambda_- t}, \tag{3.13}$$

where constants $c_\pm$ are determined by initial conditions, while the so-called *characteristic exponents* $\lambda_\pm$ are completely defined by the equation itself. In order to find the exponents, it is sufficient to plug just one partial solution, $\exp\{\lambda t\}$, into the equation. In our simple case (12), this yields the following *characteristic equation*:

$$m_{ef}\lambda^2 + \kappa_{ef} = 0. \tag{3.14}$$

If the ratio k_{ef}/m_{ef} is positive,[2] i.e. the fixed point corresponds to the minimum of potential energy (e.g., points q_0 and q_2 in Fig. 1), the characteristic equation yields

$$\lambda_\pm = \pm i\omega_0, \quad \omega_0 \equiv \left(\frac{\kappa_{ef}}{m_{ef}}\right)^{1/2}, \tag{3.15}$$

(where i is the imaginary unity, $i^2 = -1$), so that Eq. (13) describes sinusoidal oscillations of the system,

$$\tilde{q}(t) = c_+ e^{+i\omega_0 t} + c_- e^{-i\omega_0 t} = c_c \cos\omega_0 t + c_s \sin\omega_0 t, \tag{3.16}$$

with *eigenfrequency* (or "own frequency") ω_0, about the fixed point which is thereby *stable*. On the other hand, at the potential energy maximum ($k_{ef} < 0$, e.g., at point q_1 in Fig. 1), we get

$$\lambda_\pm = \pm\lambda, \quad \lambda \equiv \left(\frac{|\kappa_{ef}|}{m_{ef}}\right)^{1/2}, \quad \tilde{q}(t) = c_+ e^{+\lambda t} + c_- e^{-\lambda t}. \tag{3.17}$$

Since the solution has an exponentially growing part,[3] the fixed point is *unstable*.

---

[1] Those terms may be important only in the very special case then κ_{ef} is exactly zero, i.e. when a fixed point is an *inflection point* of function $U_{ef}(q)$.

[2] In what follows, I will assume that the effective mass m_{ef} is positive, which is true in most (but not all!) dynamic systems. The changes necessary if it is negative are obvious.

Note that the *quadratic* expansion of function $U_{ef}(q)$, given by Eq. (10), is equivalent to a *linear* expansion of the effective force:

$$F_{ef} \equiv -\frac{dU_{ef}}{dq}\Big|_{q=q_n} \approx -\kappa_{ef}\tilde{q}, \qquad (3.18)$$

immediately resulting in the linear equation (12). Hence, in order to analyze the stability of a fixed point q_n, it is sufficient to *linearize* the equation of motion in small deviations from that point, and study possible solutions of the resulting linear equation.

As an example, let us return to our testbed problem (Fig. 2.1) whose function U_{ef} we already know – see the second of Eqs. (6). With it, the equation of motion (4) becomes

$$mR^2\ddot{\theta} = -\frac{dU_{ef}}{d\theta} = mR^2\left[\omega^2\cos\theta - \Omega^2\right]\sin\theta, \quad \text{i.e. } \ddot{\theta} = \left[\omega^2\cos\theta - \Omega^2\right]\sin\theta, \qquad (3.19)$$

where $\Omega \equiv (g/R)^{1/2}$ is the frequency of small oscillations of the system at $\omega = 0$ - see Eq. (2.26).[4] From requirement (8), we see that on any 2π-long segment of angle θ,[5] the system may have four fixed points:

$$\theta_0 = 0, \qquad \theta_1 = \pi, \qquad \theta_{2,3} = \pm\cos^{-1}\frac{\Omega^2}{\omega^2}, \qquad (3.20)$$

The last two fixed points, corresponding to the bead rotating on either side of the ring, exist only if the angular velocity ω of ring rotation exceeds Ω. (In the limit of very fast rotation, $\omega >> \Omega$, Eq. (20) yields $\theta_{2,3} \rightarrow \pm\pi/2$, i.e. the stationary positions approach the horizontal diameter of the ring - in accordance with physical intuition.)

In order to analyze the fixed point stability, similarly to Eq. (9), we plug $\theta = \theta_n + \tilde{\theta}$ into Eq. (19) and Taylor-expand the trigonometric functions of θ up to the first term in $\tilde{\theta}$:

$$\ddot{\tilde{\theta}} = \left[\omega^2\left(\cos\theta_n - \sin\theta_n\,\tilde{\theta}\right) - \Omega^2\right]\left(\sin\theta_n + \cos\theta_n\,\tilde{\theta}\right). \qquad (3.21)$$

Generally, this equation may be linearized further by purging its right-hand part of the term proportional to $\tilde{\theta}^2$; however in this simple case, Eq. (21) is already convenient for analysis. In particular, for the fixed point $\theta_0 = 0$ (corresponding to the bead position at the bottom of the ring), we have $\cos\theta_0 = 1$ and $\sin\theta_0 = 0$, so that Eq. (21) is reduced to a linear differential equation

$$\ddot{\tilde{\theta}} = \left(\omega^2 - \Omega^2\right)\tilde{\theta}\ , \qquad (3.22)$$

whose characteristic equation is similar to Eq. (14) and yields

$$\lambda^2 = \omega^2 - \Omega^2, \quad \text{for } \theta \approx \theta_0. \qquad (3.23a)$$

---

[3] Mathematically, the growing part vanishes at some special (exact) initial conditions which give $c_+ = 0$. However, the futility of this argument for real physical systems should be obvious for anybody who had ever tried to balance a pencil on its sharp point.

[4] Note that Eq. (19) coincides with Eq. (2.25). This is a good sanity check illustrating that the procedure (5)-(6) of moving of a term from the potential to kinetic energy within the Lagrangian function is indeed legitimate.

[5] For this particular problem, the values of θ that differ by a multiple of 2π, are physically equivalent.

This result shows that if $\omega < \Omega$, when both roots λ are imaginary, this fixed point is stable. However, the roots become real, $\lambda_{\pm} = (\omega^2 - \Omega^2)^{1/2}$, with one of them positive, so that the fixed point becomes unstable beyond this threshold, i.e. as soon as fixed points $\theta_{2,3}$ exist. An absolutely similar calculations for other fixed points yield

$$\lambda^2 = \Omega^2 + \omega^2 > 0, \quad \text{for } \theta \approx \theta_1, \tag{3.23b}$$

$$\lambda^2 = \Omega^2 - \omega^2, \quad \text{for } \theta \approx \theta_{2,3}. \tag{3.23c}$$

These results show that fixed point θ_1 (bead on the top of the ring) is always unstable – just as we could foresee, while the side fixed points $\theta_{2,3}$ are stable as soon as they exist (at $\omega > \Omega$).

Thus, our fixed-point analysis may be summarized in a simple way: an increase of the ring rotation speed ω beyond a certain threshold value, equal to Ω (2.26), causes the bead to move on one of the ring sides, oscillating about one of the fixed points $\theta_{2,3}$. Together with the rotation about the vertical axis, this motion yields quite a complex spatial trajectory as observed from a lab frame, so it is fascinating that we could analyze it qualitatively in such a simple way.

Later in this course we will repeatedly use the linearization of the equations of motion for the analysis of stability of more complex systems, including those with energy dissipation.

3.3. Hamiltonian 1D systems

The autonomous systems that are described by time-independent Lagrangians, are frequently called *Hamiltonian,* because their Hamiltonian function H (again, not necessarily equal to the genuine mechanical energy E!) is conserved. In our current 1D case, described by Eq. (3),

$$H = \frac{m_{\text{ef}}}{2}\dot{q}^2 + U_{\text{ef}}(q) = \text{const}. \tag{3.24}$$

This is the first integral of motion. Solving Eq. (24) for $\dot{q}$, we get the first-order differential equation,

$$\frac{dq}{dt} = \pm\left\{\frac{2}{m_{\text{ef}}}\left[H - U_{\text{ef}}(q)\right]\right\}^{1/2}, \tag{3.25}$$

which may be readily integrated:

$$\pm\left(\frac{m_{\text{ef}}}{2}\right)^{1/2}\int_{q(t_0)}^{q(t)}\frac{dq'}{\left[H - U_{\text{ef}}(q')\right]^{1/2}} = t - t_0. \tag{3.26}$$

Since constant H (as well as the proper sign before the integral – see below) is fixed by initial conditions, Eq. (26) gives the reciprocal form, $t = t(q)$, of the desired law of system motion, $q(t)$. Of course, for any particular problem the integral in Eq. (26) still has to be worked out, either analytically or numerically, but even the latter procedure is typically much easier than the numerical integration of the initial, second-order differential equation of motion, because at addition of many values (to which the numerical integration is reduced[6]) the rounding errors are effectively averaged out.

---

[6] See, e.g., MA Eqs. (5.2) and (5.3).

Moreover, Eqs. (24)-(25) also allow a general classification of 1D system motion. Indeed:

(i) If $H > U_{ef}(q)$ in the whole range of interest, the effective kinetic energy T_{ef} (3) is always positive. Hence derivative dq/dt cannot change sign, so that the effective velocity retains the sign it had initially. This is the unbound motion in one direction (Fig. 2a).

(ii) Now let the particle approach a *classical turning point A* where $H = U_{ef}(x)$ - see Fig. 2b.[7] According to Eqs. (25), (26), at that point the particle velocity vanishes, while its acceleration, according to Eq. (4), is still finite. Evidently, this corresponds to the particle reflection from the "potential wall", with the change of velocity sign.

(iii) If, after the reflection from point *A*, the particle runs into another classical turning point *B* (Fig. 2c), the reflection process is repeated again and again, so that the particle is bound to a periodic motion between two turning points.

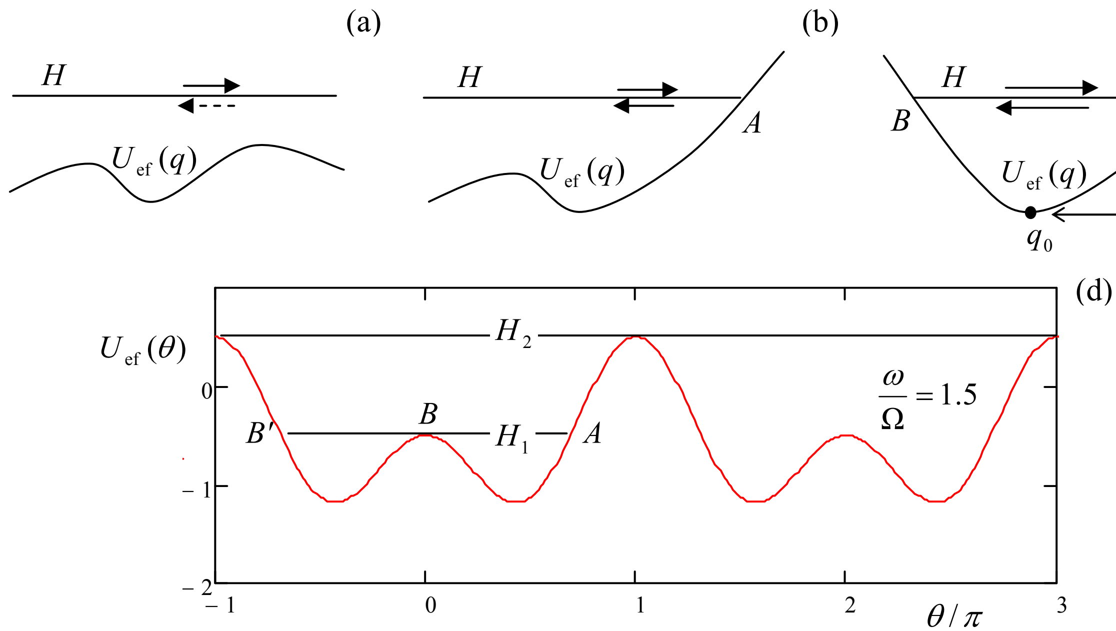


Fig. 3.2. Graphical representation of Eq. (25) for three different cases: (a) unbound motion, with the velocity sign conserved, (b) reflection from the "classical turning point", accompanied with the velocity sign change, and (c) bound, periodic motion between two turning points – schematically. (d) Effective potential energy (6) of the bead on the rotating ring (Fig. 1.5) for $\omega > \Omega$, in units of $2mgR$.

The last case of *periodic oscillations* presents large practical interest, and the whole next chapter will be devoted to a detailed analysis of this phenomenon and numerous associated effects. Here I will only note that Eq. (26) immediately enables us to calculate the oscillation period:

$$\mathcal{T} = 2\left(\frac{m_{ef}}{2}\right)^{1/2} \int_B^A \frac{dq}{[H - U_{ef}(q)]^{1/2}}, \quad (3.27)$$

Oscillation period

[7] This terminology comes from quantum mechanics which shows that actually a particle (or rather its wavefunction) can, to a certain extent, penetrate the "classically forbidden range" where $H < U_{ef}(x)$.

where the additional upfront factor 2 accounts for two time intervals: for the motion from B to A and back (Fig. 2c). Indeed, according to Eq. (25), in each classically allowed point q the velocity magnitude is the same, so that these time intervals are equal to each other.[8]

Now let us link Eq. (27) to the fixed point analysis carried out in the previous section. As Fig. 2c shows, if H is reduced to approach $U_{\min}$, the oscillations described by Eq. (27) take place at the very bottom of "potential well", about a stable fixed point q_0. Hence, if the potential energy profile is smooth enough, we may limit the Taylor expansion (10) by the quadratic term. Plugging it into Eq. (27), and using the mirror symmetry of this particular problem about the fixed point q_0, we get

$$\mathcal{T} = 4\left(\frac{m_{\text{ef}}}{2}\right)^{1/2}\int_0^A \frac{d\tilde{q}}{\left[H-\left(U_{\min}+\kappa_{\text{ef}}\tilde{q}^2/2\right)\right]^{1/2}} = \frac{4}{\omega_0}I, \quad \text{with } I \equiv \int_0^1 \frac{d\xi}{\left(1-\xi^2\right)^{1/2}}, \qquad (3.28)$$

where $\xi \equiv \tilde{q}/A$, with $A \equiv (2/\kappa_{\text{ef}})^{1/2}[H - U_{\min}]^{1/2}$ being the classical turning point, i.e. the oscillation amplitude, and ω_0 is the eigenfrequency given by Eq. (15). Taking into account that the elementary integral I in that equation equals $\pi/2$,[9] we finally get

$$\mathcal{T} = \frac{2\pi}{\omega_0}, \qquad (3.29)$$

as it should be for harmonic oscillations (16). Note that the oscillation period does not depend on the oscillation amplitude A, i.e. on the difference (H - $U_{\min}$) - while it is small.

3.4. Planetary problems

Leaving a more detailed study of oscillations for the next chapter, let us now discuss the so-called *planetary systems*[10] whose description, somewhat surprisingly, may be also reduced to an effectively 1D problem. Consider two particles that interact via a conservative, central force $\mathbf{F}_{21} = - \mathbf{F}_{12} = \mathbf{n}_r F(r)$, where r and $\mathbf{n}_r$ are, respectively, the magnitude and direction of the *distance vector* $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ connecting the two particles (Fig. 3).

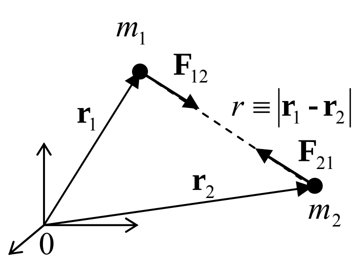


Fig. 3.3. Vectors in the "planetary" problem.

[8] Note that the dependence of points A and B on the "energy" H is not necessarily continuous. For example, for our testbed problem, whose effective potential energy is plotted in Fig. 2d (for a particular value of $\omega > \Omega$), a gradual increase of H leads to a sudden jump, at $H = H_1$, of point B to position B', corresponding to a sudden switch from oscillations about one fixed point $\theta_{2,3}$ to oscillations about two adjacent fixed points (before the beginning of a persistent rotation along the ring at $H > H_2$).

[9] Introducing a new variable ζ by relation $\xi \equiv \sin\zeta$, we get $d\xi = \cos\zeta\, d\zeta = (1 - \xi^2)^{1/2}\, d\zeta$, so that the function under the integral is just $d\zeta$.

[10] This name is very conditional, because this group of problems includes, for example, charged particle scattering (see Sec. 3.7 below).

Generally, two particles moving without constrains in 3D space, have 3 + 3 = 6 degrees of freedom that may described, e.g., by their Cartesian coordinates $\{x_1, y_1, z_1, x_2, y_2, z_2\}$ However, for this particular form of interaction, the following series of tricks allows the number of essential degrees of freedom to be reduced to just one.

First, the central, conservative force of particle interaction may be described by time-independent potential energy $U(r)$. Hence the Lagrangian of the system is

$$L \equiv T - U(r) = \frac{m_1}{2}\dot{\mathbf{r}}_1^2 + \frac{m_2}{2}\dot{\mathbf{r}}_2^2 - U(r). \tag{3.30}$$

Let us perform the transfer from the initial six scalar coordinates of the particles to six generalized coordinates: three Cartesian components of the distance vector

$$\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2, \tag{3.31}$$

and three components of vector

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{M}, \quad M \equiv m_1 + m_2, \tag{3.32}$$

Center of mass

which defines the position of the *center of mass* of the system. Solving the system of two linear equations (31) and (32) for the $\mathbf{r}_1$ and $\mathbf{r}_2$, we get

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}, \quad \mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}. \tag{3.33}$$

Plugging these relations into Eq. (30), we may reduce it to

$$L = \frac{M}{2}\dot{\mathbf{R}}^2 + \frac{m}{2}\dot{\mathbf{r}}^2 - U(r), \tag{3.34}$$

where m is the so-called *reduced mass*:

$$m \equiv \frac{m_1 m_2}{M}, \quad \text{so that } \frac{1}{m} \equiv \frac{1}{m_1} + \frac{1}{m_2}. \tag{3.35}$$

Reduced mass

Note that according to Eq. (35), the reduced mass is lower than that of the lightest component of the two-body system. If one of $m_{1,2}$ is much less that is counterpart (like it is in most star-planet or planet-satellite systems), then with a good precision $m \cong \min[m_1, m_2]$.

Since the Lagrangian function (34) depends only on $\dot{\mathbf{R}}$ rather than $\mathbf{R}$ itself, according to our discussion in Sec. 2.4, the Cartesian components of R are cyclic coordinates, and the corresponding generalized momenta are conserved:

$$P_j \equiv \frac{\partial L}{\partial \dot{R}_j} = M\dot{R}_j = \text{const}, \quad j = 1, 2, 3. \tag{3.36}$$

Physically, this is just the conservation law for the full momentum $\mathbf{P} \equiv M\dot{\mathbf{R}}$ of our system, due to absence of external forces. Actually, in the axiomatics used in Sec. 1.3 this law is postulated – see Eq. (1.10) – but now we may attribute momentum $\mathbf{P}$ to a certain geometric point, the center of mass $\mathbf{R}$. In particular, since according to Eq. (36) the center moves with constant velocity in the inertial reference frame used to write Eq. (30), we may create a new inertial frame with the origin at point **R**. In this new frame, $\mathbf{R} \equiv 0$, so that vector **r** (and hence scalar r) remain the same as in the old frame (because the frame transfer vector adds equally to $\mathbf{r}_1$ and $\mathbf{r}_2$, and cancels in $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$), and the Lagrangian (34) is now reduced to

$$L = \frac{m}{2}\dot{\mathbf{r}}^2 - U(r). \tag{3.37}$$

Thus our initial problem has been reduced to just three degrees of freedom - three scalar components of vector **r**. Moreover, Eq. (37) shows that dynamics of vector **r** of our initial, two-particle system is identical to that of the radius-vector of a *single particle* with the effective mass m, moving in the central potential field $U(r)$.

3.5. 2nd Kepler law

Two more degrees of freedom may be excluded from the planetary problem by noticing that according to Eq. (1.35), the angular momentum $\mathbf{L} = \mathbf{r}\times\mathbf{p}$ of our effective particle is also conserved, both in magnitude and direction. Since the direction of **L** is, by its definition, perpendicular to both of **r** and $\mathbf{v} = \mathbf{p}/m$, this means that particle's motion is confined to a plane (whose orientation in space is determined by the initial directions of vectors **r** and **v**). Hence we can completely describe particle's position by just two coordinates in that plane, for example by distance r to the center, and the polar angle φ. In these coordinates, Eq. (37) takes the form identical to Eq. (2.49):

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - U(r). \tag{3.38}$$

Moreover, the latter coordinate, polar angle φ, may be also eliminated by using the conservation of angular momentum's magnitude, in the form of Eq. (2.50): [11]

$$L_z = mr^2\dot{\varphi} = \text{const}. \tag{3.39}$$

A direct corollary of this conservation is the so-called *2nd Kepler law*:[12] the radius-vector **r** sweeps equal areas A in equal times. Indeed, in the linear approximation in $dA << A$, the area differential dA equals to the area of a narrow right triangle with the base being the arc differential $rd\varphi$, and the height equal to r - see Fig. 4. As a result, according to Eq. (39), the time derivative of the area,

$$\frac{dA}{dt} = \frac{r(rd\varphi)/2}{dt} = \frac{1}{2}r^2\dot{\varphi} = \frac{L_z}{2m}, \tag{3.40}$$

remains constant. Integration of this equation over an arbitrary (not necessarily small!) time interval proves the 2nd Kepler law.

[11] Here index z stands for the coordinate perpendicular to the motion plane. Since other components of the angular momentum are equal zero, the index is not really necessary, but I will still use it, just to make a clear distinction between the angular momentum L_z and the Lagrangian function L.

[12] One of three laws deduced almost exactly 400 years ago by J. Kepler (1571 – 1630), from the extremely detailed astronomical data collected by T. Brahe (1546-1601). In turn, the set of three Kepler laws were the main basis for Isaac Newton's discovery of the gravity law (1.16). That's how physics marched on…

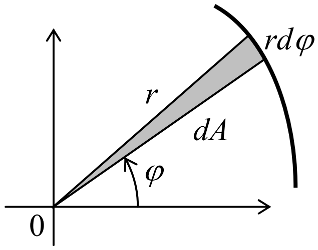


Fig. 3.4. Area differential in the polar coordinates.

Now note that since $\partial L / \partial t = 0$, the Hamiltonian function H is also conserved, and since, according to Eq. (38), the kinetic energy of the system is a quadratic-homogeneous function of the generalized velocities $\dot{r}$ and $\dot{\varphi}$, $H = E$, so that the system energy E,

$$E = \frac{m}{2}\dot{r}^2 + \frac{m}{2}r^2\dot{\varphi}^2 + U(r), \tag{3.41}$$

is also a first integral of motion.[13] But according to Eq. (39), the second term of Eq. (41) may be presented as

$$\frac{m}{2}r^2\dot{\varphi}^2 = \frac{L_z^2}{2mr^2}, \tag{3.42}$$

so that energy (41) may be expressed as that of a 1D particle moving along axis r,

$$E = \frac{m}{2}\dot{r}^2 + U_{\text{ef}}(r), \tag{3.43}$$

in the following effective potential:

$$U_{\text{ef}}(r) \equiv U(r) + \frac{L_z^2}{2mr^2}. \tag{3.44}$$

Effective potential energy

So the planetary motion problem has been reduced to the dynamics of an effectively 1D system.[14]

Now we may proceed just like we did in Sec. 3, with due respect for the very specific effective potential (44) which, in particular, diverges at $r \to 0$ - possibly besides the very special case of an exactly radial motion, $L_z = 0$. In particular, we may solve Eq. (43) for dr/dt to get

$$dt = \left(\frac{m}{2}\right)^{1/2} \frac{dr}{[E - U_{\text{ef}}(r)]^{1/2}}. \tag{3.45}$$

The integration of this relation allows us not only to get a direct relation between time t and distance r, similar to Eq. (26),

[13] One may claim that this fact should have been evident from the very beginning, because the effective particle of mass m moves in a potential field $U(r)$ which conserves energy.

[14] Note that this reduction has been done in a way different from that used for our testbed problem (shown in Fig. 2.1) in Sec. 2 above. (The reader is encouraged to analyze this difference.) In order to emphasize this fact, I will keep writing E instead of H here, though for the planetary problem we are discussing now these two notions coincide.

$$t = \pm\left(\frac{m}{2}\right)^{1/2}\int\frac{dr}{[E-U_{\text{ef}}(r)]^{1/2}} = \pm\left(\frac{m}{2}\right)^{1/2}\int\frac{dr}{[E-U(r)-L_z^2/2mr^2]^{1/2}}, \tag{3.46}$$

but also do a similar calculation of angle φ. Indeed, integrating Eq. (39),

$$\varphi \equiv \int\dot{\varphi}\,dt = \frac{L_z}{m}\int\frac{dt}{r^2}. \tag{3.47}$$

and plugging dt from Eq. (45), we get an explicit expression for particle's trajectory $\varphi(r)$:

$$\varphi = \pm\frac{L_z}{(2m)^{1/2}}\int\frac{dr}{r^2[E-U_{\text{ef}}(r)]^{1/2}} = \pm\frac{L_z}{(2m)^{1/2}}\int\frac{dr}{r^2[E-U(r)-L_z^2/2mr^2]^{1/2}}. \tag{3.48}$$

Note that according to Eq. (39), derivative $d\varphi/dt$ does *not* change sign at the reflection from any classical turning point $r \neq 0$, so that, in contrast to Eq. (46), the sign in the right-hand part of Eq. (48) is uniquely determined by the initial conditions and cannot change during the motion.

Let us use these results, valid for any interaction law $U(r)$, for the planetary motion's classification. The following cases should be distinguished. (Following a good tradition, in what follows I will select the arbitrary constant in the potential energy in the way to provide $U_{\text{ef}} \to 0$ at $r \to \infty$.)

If the particle interaction is *attractive*, and the divergence of the attractive potential at $r \to 0$ is faster than $1/r^2$, then $U_{\text{ef}}(r) \to -\infty$ at $r \to 0$, so that at appropriate initial conditions ($E < 0$) the particle may drop on the center even if $L_z \neq 0$ – the event called the *capture*. On the other hand, with $U(r)$ either converging or diverging slower than $1/r^2$ at $r \to 0$, the effective energy profile $U_{\text{ef}}(r)$ has the shape shown schematically in Fig. 5. This is true, in particular, for the very important case

Attractive Coulomb potential

$$U(r) = -\frac{\alpha}{r}, \quad \alpha > 0, \tag{3.49}$$

which describes, in particular, the *Coulomb* (electrostatic) *interaction* of two particles with electric charges of the opposite sign, and Newton's gravity law (1.16a). This particular case will be analyzed in the following section, but now let us return to the analysis of an arbitrary attractive potential $U(r) < 0$ leading to the effective potential shown in Fig. 5, when the angular-momentum term dominates at small distances r.

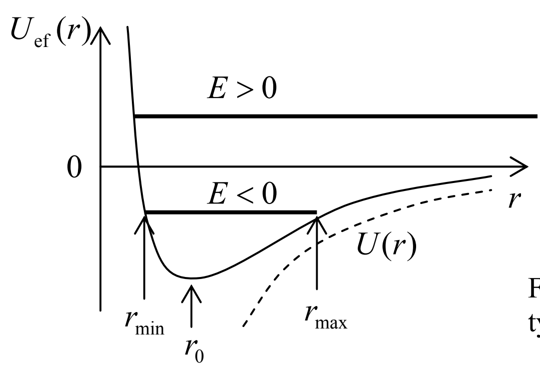


Fig. 3.5. Effective potential profile of, and two types of motion in an attractive central field.

According to the analysis of Sec. 3, such potential profile, with a minimum at some distance r_0, may sustain two types of motion, depending on the energy E (which is of course determined by the initial conditions):

(i) If $E > 0$, there is only one classical turning point where $E = U_{ef}$, so that distance r either grows with time from the very beginning, or (if the initial value of $\dot{r}$ was negative) first decreases and then, after the reflection from the increasing potential U_{ef}, starts to grow indefinitely. The latter case, of course, describes *scattering*.

(ii) On the opposite, if the energy is within the range

$$U_{ef}(r_0) \leq E < 0, \tag{3.50}$$

the system moves periodically between two classical turning points r_{min} and r_{max}. These oscillations of distance r correspond to the bound orbital motion of our effective particle about the attracting center.[15]

Let us start with the discussion of the bound motion, with energy within the range (50). If energy has its minimal possible value,

$$E = U_{ef}(r_0) = \min[U_{ef}(r)], \tag{3.51}$$

the distance cannot change, $r = r_0 =$ const, so that the orbit is circular, with the radius r_0 satisfying the condition $dU_{ef}/dr = 0$. Let us see whether this result allows for an elementary explanation. Using Eq. (44) we see that the condition for r_0 may be written as

$$\frac{L_z^2}{mr_0^3} = \frac{dU}{dr}\bigg|_{r=r_0}. \tag{3.52}$$

Since in a circular motion, velocity **v** is perpendicular to the radius vector **r,** L_z is just mr_0v, the left-hand part of Eq. (52) equals mv^2/r_0, while its right-hand part is just the magnitude of the attractive force, so that this equation expresses the well-known 2nd Newton law for the circular motion. Plugging this result into Eq. (47), we get a linear law of angle change, $\varphi = \omega t + \text{const}$, with angular velocity

$$\omega = \frac{L_z}{mr_0^2} = \frac{v}{r_0}, \tag{3.53}$$

and hence the rotation period $\mathcal{T}_\varphi \equiv 2\pi/\omega$ obeys the elementary relation

$$\mathcal{T}_\varphi = \frac{2\pi r_0}{v}. \tag{3.54}$$

Now, let the energy be above its minimum value. Using Eq. (46) just as in Sec. 3, we see that distance r now oscillates with period

$$\mathcal{T}_r = (2m)^{1/2} \int_{r_{min}}^{r_{max}} \frac{dr}{[E - U(r) - L_z^2/2mr^2]^{1/2}}. \tag{3.55}$$

[15] In the opposite case when the interaction is *repulsive*, $U(r) > 0$, the addition of the positive angular energy term only increases the trend, and only the scattering scenario is possible.

This period is, in general, different from $\mathcal{T}_\varphi$. Indeed, the change of angle φ between two sequential points of the nearest approach, that follows from Eq. (48),

$$\left|\Delta\varphi\right| = 2\frac{L_z}{(2m)^{1/2}}\int_{r_{\min}}^{r_{\max}}\frac{dr}{r^2\left[E - U(r) - L_z^2/2mr^2\right]^{1/2}}, \tag{3.56}$$

is generally different from 2π. Hence, the general trajectory of the bound motion has a spiral shape – see, e.g., an illustration in Fig. 6.

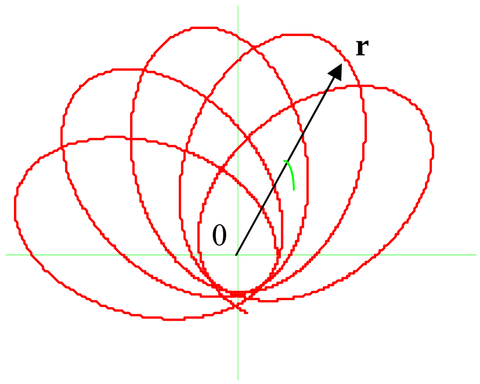


Fig. 3.6. Typical open orbit of a particle moving in a non-Coulomb central field.

3.6. 1st and 3rd Kepler laws

The situation is special, however, for a very important particular case, namely that of the *Coulomb potential* described by Eq. (49). Indeed, plugging this potential into Eq. (48), we get

$$\varphi = \pm\frac{L_z}{(2m)^{1/2}}\int\frac{dr}{r^2\left(E + \alpha/r - L_z^2/2mr^2\right)^{1/2}}. \tag{3.57}$$

This is a table integral,[16] equal to

$$\varphi = \pm\cos^{-1}\frac{L_z^2/m\alpha r - 1}{\left(1 + 2EL_z^2/m\alpha^2\right)^{1/2}} + \text{const.} \tag{3.58}$$

The reciprocal function, $r(\varphi)$, is 2π- periodic:

$$r = \frac{p}{1 + e\cos(\varphi + \text{const})}, \tag{3.59}$$

Elliptic orbit and its parameters

so that at $E < 0$, the orbit a closed line,[17] characterized with the following parameters:

$$p \equiv \frac{L_z^2}{m\alpha}, \qquad e \equiv \left[1 + \frac{2EL_z^2}{m\alpha^2}\right]^{1/2}. \tag{3.60}$$

[16] See, e.g., MA Eq. (6.3a).

[17] It may be proved that for the power-law interaction, $U \propto r^\nu$, the orbits are closed line only if $\nu = -1$ (i.e. our current case of the Coulomb potential) or $\nu = +2$ (the 3D harmonic oscillator) – the so-called *Bertrand theorem*.

The physical meaning of these parameters is very simple. Indeed, according to the general Eq. (52), in the Coulomb potential, for which $dU/dr = \alpha/r^2$, we see that p is just the circular orbit radius[18] for given L_z: $r_0 = L_z^2/m\alpha \equiv p$, and

$$\min[U_{\text{ef}}(r)] \equiv U_{\text{ef}}(r_0) = -\frac{\alpha^2 m}{2L_z^2}, \tag{3.61}$$

Using this equality, parameter e (called *eccentricity*) may be presented just as

$$e = \left\{1 - \frac{E}{\min[U_{\text{ef}}(r)]}\right\}^{1/2}. \tag{3.62}$$

Analytical geometry tells us that Eq. (59), with $e < 1$, is one of canonical forms for presentation of an *ellipse,* with one of its two focuses located at the origin. This fact is known as the *1st Kepler law*. Figure 7 shows the relation between the main dimensions of the ellipse and parameters p and e.[19]

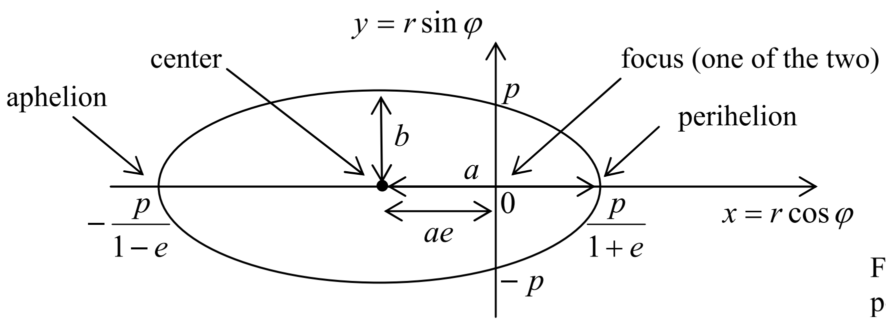


Fig. 3.7. Ellipse, and its special points and dimensions.

In particular, the major axis a and minor axis b are simply related to p and e and hence, via Eqs. (60), to the motion integrals E and L_z:

$$a = \frac{p}{1-e^2} = \frac{\alpha}{2|E|}, \qquad b = \frac{p}{\left(1-e^2\right)^{1/2}} = \frac{L_z}{\left(2m|E|\right)^{1/2}}. \tag{3.63}$$

As was mentioned above, at $E \to \min[U_{\text{ef}}(r)]$ the orbit is almost circular, with $r(\varphi) \cong r_0 = p$. On the contrary, as E is increased to approach zero (its maximum value for the closed orbit), then $e \to 1$, so that the aphelion point $r_{\max} = p/(1 - e)$ tends to infinity, i.e. the orbit becomes extremely extended. If the energy is exactly zero, Eq. (59) (with $e = 1$) is still valid for all values of φ (except for one special point $\varphi = \pi$ where r becomes infinite) and describes a *parabolic* (i.e. open) trajectory. At $E > 0$, Eq. (59) is still valid within a certain sector of angles φ (in that it yields positive results for r), and describes an open, *hyperbolic* trajectory - see the next section.

For $E < 0$, the above relations also allow a ready calculation of the rotation period $\mathcal{T} \equiv \mathcal{T}_r = \mathcal{T}_\varphi$. . (In the case of a closed trajectory, $\mathcal{T}_r$ and $\mathcal{T}_\varphi$ have to coincide.) Indeed, it is well known that the ellipse area $A = \pi ab$. But according to the 2nd Kepler law (40), $dA/dt = L_z/2m = \text{const}$. Hence

[18] Mathematicians prefer a more solemn terminology: parameter $2p$ is called the *latus rectum* of the elliptic trajectory – see Fig. 7.

[19] In this figure, the constant participating in Eqs. (58)-(59) is assumed to be zero. It is evident that a different choice of the constant corresponds just to a constant turn of the ellipse about the origin.

$$\mathcal{T} = \frac{A}{dA/dt} = \frac{\pi ab}{L_z/2m}. \qquad (3.64a)$$

Using Eqs. (60) and (63), this result may be presented in several other forms:

$$\mathcal{T} = \frac{\pi p^2}{(1-e^2)^{3/2}(L_z/2m)} = \pi\alpha\left(\frac{m}{2|E|^3}\right)^{1/2} = 2\pi a^{3/2}\left(\frac{m}{\alpha}\right)^{1/2}. \qquad (3.64b)$$

Since for the Newtonian gravity (1.16a), $\alpha = Gm_1m_2 = GmM$, at $m_1 << m_2$ (i.e. $m << M$) this constant is proportional to m, and the last form of Eq. (64b) yields the *3rd Kepler law*: periods of motion of different planets in the same central field, say that of our Sun, scale as $T \propto a^{3/2}$. Note that in contrast to the 2nd Kepler law (that is valid for any central field), the 1st and 3rd Kepler laws are potential-specific.

3.7. Classical theory of elastic scattering

If $E > 0$, the motion is unbound for any interaction potential. In this case, the two most important parameters of the particle trajectory are the *scattering angle* θ and *impact parameter* b (Fig. 8), and the main task for theory is to find the relation between them in the given potential $U(r)$. For that, it is convenient to note that b is related to two conserved quantities, particle's energy[20] E and its angular momentum L_z, in a simple way:[21]

$$L_z = b(2mE)^{1/2}. \qquad (3.65)$$

Hence the angular contribution to the effective potential (44) may be presented as

$$\frac{L_z^2}{2mr^2} = E\frac{b^2}{r^2}. \qquad (3.66)$$

Second, according to Eq. (48), the trajectory sections from infinity to the nearest approach point ($r = r_{\min}$), and from that point to infinity, have to be similar, and hence correspond to equal angle changes φ_0 - see Fig. 8.

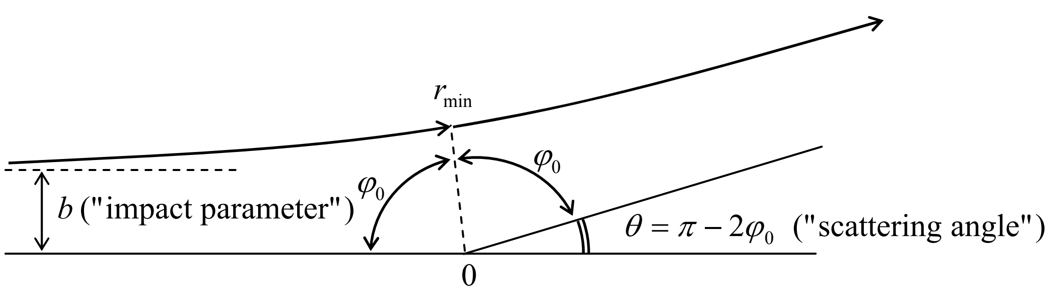


Fig. 3.8. Main geometric parameters of the scattering problem.

[20] The energy conservation law is frequently emphasized by calling this process *elastic scattering*.
[21] Indeed, at $r >> b$, the definition $\mathbf{L} = \mathbf{r}\times(m\mathbf{v})$ yields $L_z = bmv_\infty$, where $v_\infty = (2E/m)^{1/2}$ is the initial (and hence the final) velocity of the particle.

Hence we may apply the general Eq. (48) to just one of the sections, say [r_{min}, ∞], to find the scattering angle:

$$\theta = \pi - 2\varphi_0 = \pi - 2\frac{L_z}{(2m)^{1/2}}\int_{r_{min}}^{\infty}\frac{dr}{r^2\left[E - U(r) - L_z^2/2mr^2\right]^{1/2}} = \pi - 2\int_{r_{min}}^{\infty}\frac{bdr}{r^2\left[1 - U(r)/E - b^2/r^2\right]^{1/2}}. \quad (3.67)$$

In particular, for the Coulomb potential (49), now with an arbitrary sign of α, we can apply the same table integral as in the previous section to get[22]

$$\theta = \left|\pi - 2\cos^{-1}\frac{\alpha/2Eb}{\left[1 + (\alpha/2Eb)^2\right]^{1/2}}\right|. \quad (3.68a)$$

This result may be more conveniently rewritten as

$$\tan\frac{|\theta|}{2} = \frac{|\alpha|}{2Eb}. \quad (3.68b)$$

Very clearly, the scattering angle's magnitude increases with the potential strength α, and decreases as either the particle energy or the impact parameter (or both) are increased.

The general equation (67) and the Coulomb-specific relations (68) present a formally complete solution of the scattering problem. However, in a typical experiment on elementary particle scattering the impact parameter b of a single particle is random and unknown. In this case, our results may be used to obtain statistics of the scattering angle θ, in particular the so-called *differential cross-section*[23]

Differential cross-section

$$\frac{d\sigma}{d\Omega} \equiv \frac{1}{n}\frac{dN}{d\Omega}, \quad (3.69)$$

where n is the average number of the incident particles per unit area, and dN is the average number of particles scattered into a small solid angle range $d\Omega$. For a spherically-symmetric scattering center, which provides an axially-symmetric scattering pattern, $d\sigma/d\Omega$ may be calculated by counting the number of incident particles within a small range db of the impact parameter:

$$dN = n2\pi bdb. \quad (3.70)$$

and hence scattered into the corresponding small solid angle range $d\Omega = 2\pi\sin\theta\, d\theta$. Plugging these relations into Eq. (69), we get the following general geometric relation:

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|. \quad (3.71)$$

In particular, for the Coulomb potential (49), a straightforward differentiation of Eq. (68) yields the so-called *Rutherford scattering formula*

[22] Alternatively, this result may be recovered directly from Eq. (59) whose parameters, at $E > 0$, may be expressed via the same dimensionless parameter $(2Eb/\alpha)$: $p = b(2Eb/\alpha)$, $e = [1 + (2Eb/\alpha)^2]^{1/2} > 1$.

[23] This terminology stems from the fact that an integral of $d\sigma/d\Omega$ over the full solid angle, called the *full cross-section* σ, has the dimension of area: $\sigma = N/n$, where N is the total number of scattered particles.

Rutherford scattering formula

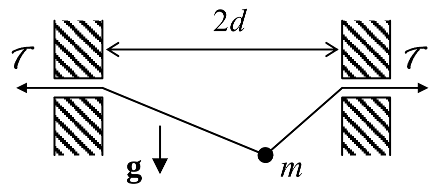
$$\frac{d\sigma}{d\Omega} = \left(\frac{\alpha}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}. \quad (3.72)$$

This result, which shows very strong scattering to small angles (so strong that the integral that expresses the full cross-section σ is formally diverging at $\theta \to 0$),[24] and weak *backscattering* (scattering to angles $\theta \approx \pi$) was historically extremely significant: in the early 1910s its good agreement with α-particle scattering experiments carried out by E. Rutherford's group gave a strong justification for "planetary" models of atoms, with electrons moving about very small nuclei.

Note that elementary particle scattering is frequently accompanied with electromagnetic radiation and/or other processes leading to the loss of the initial mechanical energy of the system, leading to *inelastic scattering*, that may give significantly different results. (In particular, a capture of an incoming particle becomes possible even for a Coulomb attracting center.) Also, quantum-mechanical effects may be important at scattering, so that the above results should be used with caution.

3.8. Exercise problems

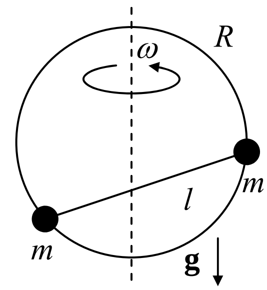
3.1. For the system considered in Problem 2.5 (a bead sliding along a string with fixed tension $\mathscr{T}$, see Fig. on the right), analyze small oscillations of the bead near the equilibrium.



3.2. Calculate the functional dependence of period $\mathscr{T}$ of oscillations of a 1D particle of mass m in potential $U(q) = \alpha q^{2n}$ (where $\alpha > 0$, and n is a positive integer) on energy E. Explore the limit $n \to \infty$.

3.3. Explain why the term $mr^2\dot{\varphi}^2/2$, recast in accordance with Eq. (42), cannot be merged with $U(r)$ in Eq. (38), to form an effective 1D potential energy $U(r) - L_z^2/2mr^2$, with the second term's sign opposite to that given by Eq. (44). We have done an apparently similar thing for our testbed, bead-on-rotating-ring problem in the very end of Sec. 1 – see Eq. (3.6); why cannot the same trick work for the planetary problem? Besides a formal explanation, discuss the physics behind this difference.

3.4. A dumbbell, consisting of two equal masses m on a light rod of length l, can slide without friction along a vertical ring of radius R, rotated about its vertical diameter with constant angular velocity ω - see Fig. on the right. Derive the condition of stability of the lower horizontal position of the dumbbell.



[24] This divergence, which persists at the quantum-mechanical treatment of the problem, is due to particles with large values of b, and disappears at an account, for example, of a finite concentration of the scattering centers.

3.5.[25] Analyze the dynamics of the so-called *spherical pendulum* - a point mass hung, in a uniform gravity field **g**, on a light cord of length l, with no motion's confinement to a vertical plane. In particular:

(i) find the integrals of motion and reduce the problem to a 1D one,
(ii) calculate the time period of the possible circular motion around the vertical axis,
(iii) explore small deviations from the circular motion. (Are the pendulum orbits closed?)

3.6. The orbits of Mars and Earth around the Sun may be well approximated as circles, with a radii ratio of 3/2. Use this fact, and the Earth year duration (which you should know :-), to calculate the time of travel to Mars spending least energy, neglecting the planets' size and the effects of their gravitational fields on the spacecraft.

3.7. Derive first-order and second-order differential equations for $u \equiv 1/r$ as a function of φ, describing the trajectory of particle's motion in a central potential $U(r)$. Spell out the latter equation for the particular case of the Coulomb potential (3.49) and discuss the result.

3.8. For motion in the central potential

$$U(r) = -\frac{\alpha}{r} + \frac{\beta}{r^2},$$

(i) find the orbit $r(\varphi)$, for positive α and β, and all possible ranges of energy E;
(ii) prove that in the limit $\beta \to 0$, and for energy $E < 0$, the orbit may be represented as a slowly rotating ellipse;
(iii) express the angular velocity of this slow orbit rotation via parameters α and β of the potential, particle's mass m, its energy E, and the angular momentum L_z.

3.9. A particle is moving in the field of an attractive central force, with potential

$$U(r) = -\frac{\alpha}{r^n}, \qquad \text{where } \alpha n > 0.$$

For what values of n is a circular orbit stable?

3.10. Determine the condition for a particle of mass m, moving under the effect of a central attractive force

$$\mathbf{F} = -\alpha \frac{\mathbf{r}}{r^3} \exp\left\{-\frac{r}{R}\right\},$$

where α and R are positive constants, to have a stable circular orbit.

3.11. A particle of mass m, with angular momentum L_z, moves in the field of an attractive central force with a distance-independent magnitude F. If particle's energy E is slightly higher than the value $E_{\min}$ corresponding to the circular orbit of the particle, what is the time period of its radial oscillations? Compare the period with that of the circular orbit at $E = E_{\min}$.

[25] Solving this problem is a very good preparation for the analysis of symmetric top rotation in Sec. 6.5.

3.12. For particle scattering in a repulsive Coulomb field, calculate the minimum approach distance r_{min} and velocity v_{min} at that point, and analyze their dependence on the impact parameter b (see Fig. 3.8 of the lecture notes) and the initial velocity v_∞ of the particle.

3.13. A particle is launched from afar, with impact parameter b, toward an attracting center with central potential

$$U(r) = -\frac{\alpha}{r^n}, \quad \text{with } n > 2, \alpha > 0.$$

(i) Express the minimum distance between the particle and the center via b, if the initial kinetic energy E of the particle is barely sufficient for escaping the capture by the attracting center.

(ii) Calculate capture's full cross-section; explore the limit $n \to 2$.

3.14. A meteorite with initial velocity v_∞ approaches an atmosphere-free planet of mass M and radius R.

(i) Find the condition on the impact parameter b for the meteorite to hit planet's surface.

(ii) If the meteorite barely avoids the collision, what is its scattering angle?

3.15. Calculate the differential and full cross-sections of the classical, elastic scattering of small particles by a hard sphere of radius R.

3.16. The most famous[26] confirmation of Einstein's general relativity theory has come from the observation, by A. Eddington and his associates, of light's deflection by the Sun, during the May 1919 solar eclipse. Considering light photons as classical particles propagating with the light speed $v_0 \to c \approx 2.998\times10^8$m/s, and the astronomic data for Sun's mass, $M_S \approx 1.99\times10^{30}$kg, and radius, $R_S \approx 0.6957\times10^9$m, calculate the nonrelativistic mechanics' prediction for the angular deflection of the light rays grazing the Sun's surface.

[26] It was not the first confirmation, though. The first one came 4 years earlier from A. Einstein himself, who showed that his theory may qualitatively explain the difference between the rate of Mercury orbit's precession, known from earlier observations, and the nonrelativistic theory of this effect.

This page is intentionally left blank