

## Chapter 2. Lagrangian Formalism

The goal of this chapter is to describe the Lagrangian formulation of analytical mechanics, which is extremely useful for obtaining the differential equations of motion (and sometimes their first integrals) not only for mechanical systems with holonomic constraints, but also other dynamic systems.

### 2.1. Lagrange equations

In many cases, the constraints imposed on 3D motion of a system of  $N$  particles may be described by  $N$  vector (i.e.  $3N$  scalar) algebraic equations

$$\mathbf{r}_k = \mathbf{r}_k(q_1, q_2, \dots, q_j, \dots, q_J, t), \quad 1 \leq k \leq N, \quad (2.1)$$

where  $q_j$  are certain *generalized coordinates* which (together with constraints) completely define the system position, and  $J \leq 3N$  is the number of the actual *degrees of freedom*. The constraints that allow such description are called *holonomic*.<sup>1</sup>

For example, for our testbed, bead-on-rotating-ring problem (see Fig. 1.5 and Fig. 1 below)  $J = 1$ , because taking into account the constraints imposed by the ring, bead's position may be uniquely determined by just one generalized coordinate – for example, its polar angle  $\theta$ . Indeed, selecting the reference frame as shown in Fig. 1 and using the well-known formulas for the spherical coordinates,<sup>2</sup> we see that in this case Eq. (1) in Cartesian coordinates has the form

$$\mathbf{r} \equiv \{x, y, z\} = \{R \sin \theta \cos \varphi, R \sin \theta \sin \varphi, R \cos \theta\}, \quad \text{where } \varphi = \omega t + \text{const}, \quad (2.2)$$

where the constant depends on the exact selection of axes  $x$  and  $y$  and the time origin. Since  $\varphi(t)$  is a fixed function of time, and  $R$  is a fixed constant, the position of particle in space at any instant  $t$  is indeed completely determined by the value of its only generalized coordinate  $\theta$ . Note that the dimensionality of the generalized coordinate may be different from that of Cartesian coordinates (meters)!

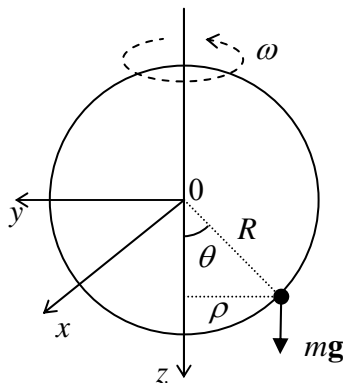


Fig. 2.1. Bead on a rotating ring as an example of the system with just one degree of freedom:  $J = 1$ .

<sup>1</sup> Possibly, the simplest example of a *non-holonomic* constraint is a set of inequalities describing the hard walls confining the motion of particles in a closed volume. Non-holonomic constraints are better dealt with other methods, e.g., by imposing proper boundary conditions on the (otherwise unconstrained) motion.

<sup>2</sup> See, e.g., MA Eq. (10.7).

Now returning to the general case of  $J$  degrees of freedom, let us consider a set of small *variations* (alternatively called “virtual displacements”)  $\delta q_j$  allowed by the constraints. Virtual displacements differ from the actual small displacements (described by *differentials*  $dq_j$  proportional to time variation  $dt$ ) in that  $\delta q_j$  describes not the system’s motion as such, but rather its possible variation – see Fig. 1.

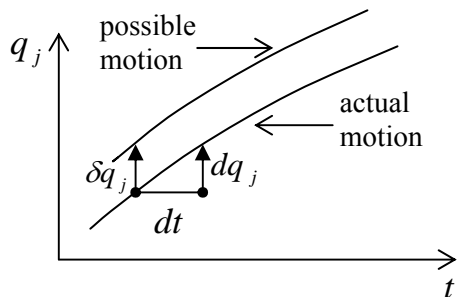


Fig. 2.2. Actual displacement  $dq_j$  vs. the virtual one (i.e. variation)  $\delta q_j$ .

Generally, operations with variations are the subject of a special field of mathematics, the calculus of variations.<sup>3</sup> However, the only math background necessary for our current purposes is the understanding that operations with variations are similar to those with the usual differentials, though we need to watch carefully what each variable is a function of. For example, if we consider the variation of the radius-vectors (1), at a fixed time  $t$ , as a function of independent variations  $\delta q_j$ , we may use the usual formula for the differentiation of a function of several arguments:<sup>4</sup>

$$\delta \mathbf{r}_k = \sum_j \frac{\partial \mathbf{r}_k}{\partial q_j} \delta q_j. \quad (2.3)$$

Now let us break the force acting upon the  $k$ -th particle into two parts: the frictionless, constraining part  $\mathbf{N}_k$  of the reaction force and the remaining part  $\mathbf{F}_k$  – including the force components from other sources and possibly the friction part of the reaction force. Then the 2<sup>nd</sup> Newton law for  $k$ -th particle of the system may be presented as

$$m_k \dot{\mathbf{v}}_k - \mathbf{F}_k = \mathbf{N}_k. \quad (2.4)$$

Since any variation of the motion has to be allowed by the constraints, its  $3N$ -dimensional vector with  $N$  3D-vector components  $\delta \mathbf{r}_k$  has to be perpendicular to the  $3N$ -dimensional vector of the constraining forces, also with  $N$  3D-vector components  $\mathbf{N}_k$ . (For example, for the problem shown in Fig. 2.1, the virtual displacement vector  $\delta \mathbf{r}_k$  may be directed only along the ring, while the constraining force  $\mathbf{N}$ , exerted by the ring, has to be perpendicular to that direction.) This condition may be expressed as

$$\sum_k \mathbf{N}_k \cdot \delta \mathbf{r}_k = 0, \quad (2.5)$$

<sup>3</sup> For a concise introduction to the field see, e.g., I. Gelfand and S. Fomin, *Calculus of Variations*, Dover, 2000 or L. Elsgolc, *Calculus of Variations*, Dover, 2007. An even shorter review may be found in Chapter 17 of Arfken and Weber - see MA Sec. 16. For a more detailed discussion, using many examples from physics, see R. Weinstock, *Calculus of Variations*, Dover, 2007.

<sup>4</sup> See, e.g., MA Eq. (4.2). In all formulas of this section, all summations over index  $j$  are from 1 to  $J$ , while those over the particle number  $k$  are from 1 to  $N$ .

where the scalar product of  $3N$ -dimensional vectors is defined exactly as that of 3D vectors, i.e. as the sum of the products of the corresponding components of the operands. The substitution of Eq. (4) into Eq. (5) results in the so-called *D'Alembert principle*:<sup>5</sup>

D'Alembert  
principle

$$\sum_k (m_k \dot{\mathbf{v}}_k - \mathbf{F}_k) \cdot \delta \mathbf{r}_k = 0. \quad (2.6)$$

Now we may plug Eq. (3) into Eq. (6) to get

$$\sum_j \left\{ \sum_k m_k \dot{\mathbf{v}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} - \mathcal{F}_j \right\} \delta q_j = 0 \quad (2.7)$$

where scalars  $\mathcal{F}_j$ , called *generalized forces*, are defined as follows:<sup>6</sup>

$$\mathcal{F}_j \equiv \sum_k \mathbf{F}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j}. \quad (2.8)$$

Now we may use the standard argument of the calculus of variations: in order for the left-hand part of Eq. (7) to be zero for an arbitrary selection of independent variations  $\delta q_j$ , the expressions in the curly brackets, for every  $j$ , should equal zero. This gives us a set of  $J$  equations

$$\sum_k m_k \dot{\mathbf{v}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} - \mathcal{F}_j = 0; \quad (2.9)$$

let us present them in a more convenient form. First, using the differentiation by parts to calculate the following time derivative:

$$\frac{d}{dt} \left( \mathbf{v}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} \right) = \dot{\mathbf{v}}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} + \mathbf{v}_k \cdot \frac{d}{dt} \left( \frac{\partial \mathbf{r}_k}{\partial q_j} \right), \quad (2.10)$$

we may notice that the first term in the right-hand part is exactly the scalar product in the first term of Eq. (9).

Second, let us use another key fact of the calculus of variations (which is, essentially, evident from Fig. 3): the differentiation of a variable over time and over the generalized coordinate variation (at fixed time) are interchangeable operations.

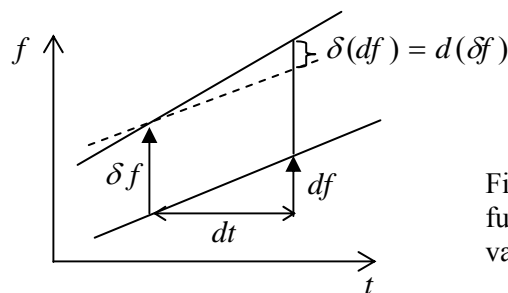


Fig. 2.3. Variation of the differential (of any function  $f$ ) equals the differential of its variation.

<sup>5</sup> It had been spelled out in a 1743 work by J.-B. le Rond d'Alembert, though the core of this result has been traced to an earlier work by J. Bernoulli (1667 – 1748).

<sup>6</sup> Note that since the dimensionality of generalized coordinates may be arbitrary, that of generalized forces may also differ from the newton.

As a result, in the second term on the right-hand part of Eq. (10) we may write

$$\frac{d}{dt} \left( \frac{\partial \mathbf{r}_k}{\partial q_j} \right) = \frac{\partial}{\partial q_j} \left( \frac{d\mathbf{r}_k}{dt} \right) \equiv \frac{\partial \mathbf{v}_k}{\partial q_j}. \quad (2.11)$$

Finally, let us differentiate of Eq. (1) over time:

$$\mathbf{v}_k \equiv \frac{d\mathbf{r}_k}{dt} = \sum_j \frac{\partial \mathbf{r}_k}{\partial q_j} \dot{q}_j + \frac{\partial \mathbf{r}_k}{\partial t}. \quad (2.12)$$

This equation shows that particle velocities  $\mathbf{v}_k$  may be considered as linear functions of the generalized velocities  $\dot{q}_j$  *considered as independent variables*, with proportionality coefficients

$$\frac{\partial \mathbf{v}_k}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_k}{\partial q_j}. \quad (2.13)$$

With the account of Eqs. (10), (11), and (13), Eq. (9) turns into

$$\frac{d}{dt} \sum_k m_k \mathbf{v}_k \cdot \frac{\partial \mathbf{v}_k}{\partial \dot{q}_j} - \sum_k m_k \mathbf{v}_k \cdot \frac{\partial \mathbf{v}_k}{\partial q_j} - \mathcal{F}_j = 0 \quad (2.14)$$

This result may be further simplified by making, for the total kinetic energy of the system,

$$T \equiv \sum_k \frac{m_k}{2} v_k^2 = \frac{1}{2} \sum_k m_k \mathbf{v}_k \cdot \mathbf{v}_k, \quad (2.15)$$

the same commitment as for  $\mathbf{v}_k$ , i.e. considering  $T$  a function of not only the generalized coordinates  $q_j$  and time  $t$ , but also of the generalized velocities  $\dot{q}_j$  - as variables independent of  $q_j$  and  $t$ . Then we may calculate the partial derivatives of  $T$  as

$$\frac{\partial T}{\partial q_j} = \sum_k m_k \mathbf{v}_k \cdot \frac{\partial \mathbf{v}_k}{\partial q_j}, \quad \frac{\partial T}{\partial \dot{q}_j} = \sum_k m_k \mathbf{v}_k \cdot \frac{\partial \mathbf{v}_k}{\partial \dot{q}_j}, \quad (2.16)$$

and notice that they are exactly the two sums participating in Eq. (14). As a result, we get a system of  $J$  *Lagrange equations*,<sup>7</sup>

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - \mathcal{F}_j = 0, \quad \text{for } j = 1, 2, \dots, J. \quad (2.17)$$

General  
Lagrange  
equations

Their big advantage over the initial Newton law equations (4) is that the Lagrange equations do not include the constraining forces  $\mathbf{N}_k$ .

This is as far as we can go for arbitrary forces. However, if all the forces may be expressed in the form similar but somewhat more general than Eq. (1.31),  $\mathbf{F}_k = -\nabla_k U(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t)$ , where  $U$  is the

<sup>7</sup> They were derived in 1788 by J.-L. Lagrange who pioneered the whole field of analytical mechanics - not to mention his key contributions to number theory and celestial mechanics.

effective potential energy of the system,<sup>8</sup> and sign  $\nabla_k$  denotes differentiation over coordinates of  $k$ -th particle, we may recast Eq. (8) into a simpler form:

$$\mathcal{F}_j \equiv \sum_k \mathbf{F}_k \cdot \frac{\partial \mathbf{r}_k}{\partial q_j} = - \sum_k \left( \frac{\partial U}{\partial x_k} \cdot \frac{\partial x_k}{\partial q_j} + \frac{\partial U}{\partial y_k} \cdot \frac{\partial y_k}{\partial q_j} + \frac{\partial U}{\partial z_i} \cdot \frac{\partial z_i}{\partial q_j} \right) \equiv - \frac{\partial U}{\partial q_j}. \quad (2.18)$$

Since we assume that  $U$  depends only on particle coordinates (and possibly time), but not velocities,  $\partial U / \partial \dot{q}_j = 0$ , with the substitution of Eq. (18), the Lagrange equation (17) may be presented in its *canonical form*

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad \text{where } L \equiv T - U. \quad (2.19a)$$

where  $L$  is called the *Lagrangian function* (or just the ‘‘Lagrangian’’), defined as

$$L \equiv T - U. \quad (2.19b)$$

It is crucial to distinguish this function from the mechanical energy (1.26),  $E = T + U$ .

Using the Lagrangian formalism in practice, the reader should always remember that:

(i) Each system has only *one* Lagrange function  $L$ , but is described by  $J \geq 1$  Lagrange equations of motion (for  $j = 1, 2, \dots, J$ ).

(ii) Differentiating  $T$ , we have to consider the generalized velocities  $\dot{q}_j$  as independent variables, ignoring the fact they are actually the time derivatives of  $q_j$ .

## 2.2. Examples

As the first, simplest example, consider a particle constrained to move along one axis (say,  $x$ ):

$$T = \frac{m}{2} \dot{x}^2, \quad U = U(x, t). \quad (2.20)$$

In this case, it is natural to consider  $x$  as the (only) generalized coordinate, and  $\dot{x}$  as the generalized velocity, so that

$$L \equiv T - U = \frac{m}{2} \dot{x}^2 - U(x, t). \quad (2.21)$$

Considering  $\dot{x}$  an independent variable, we get  $\partial L / \partial \dot{x} = m\dot{x}$ , and  $\partial L / \partial x = -\partial U / \partial x$ , so that the Lagrange equation of motion (only one equation in this case of the single degree of freedom!) yields

$$\frac{d}{dt}(m\dot{x}) - \left( - \frac{\partial U}{\partial x} \right) = 0, \quad (2.22)$$

<sup>8</sup> Note that due to the possible time dependence of  $U$ , Eq. (17) does not mean that forces  $\mathbf{F}_k$  have to be conservative – see the next section for more discussion. With this understanding, I will still use for function  $U$  the convenient name of ‘‘potential energy’’.

evidently the same result as the  $x$ -component of the 2<sup>nd</sup> Newton law with  $F_x = -\partial U/\partial x$ . This is a good sanity check, but we see that the Lagrange formalism does not provide too much advantage in this particular case.

This advantage is, however, evident for our testbed problem – see Fig. 1. Indeed, taking the polar angle  $\theta$  for the (only) generalized coordinate, we see that in this case the kinetic energy depends not only on the generalized velocity, but also on the generalized coordinate:<sup>9</sup>

$$T = \frac{m}{2}R^2(\dot{\theta}^2 + \omega^2 \sin^2 \theta), \quad U = -mgz + \text{const} = -mgR \cos \theta + \text{const}, \quad (2.23)$$

$$L \equiv T - U = \frac{m}{2}R^2(\dot{\theta}^2 + \omega^2 \sin^2 \theta) + mgR \cos \theta + \text{const}.$$

Here it is especially important to remember that at substantiating the Lagrange equation,  $\theta$  and  $\dot{\theta}$  have to be treated as independent arguments of  $L$ , so that

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = mR^2\omega^2 \sin \theta \cos \theta - mgR \sin \theta, \quad (2.24)$$

giving us the following equation of motion:

$$\frac{d}{dt}(mR^2\dot{\theta}) - (mR^2\omega^2 \sin \theta \cos \theta - mgR \sin \theta) = 0. \quad (2.25)$$

As a sanity check, at  $\omega = 0$ , Eq. (25) is reduced to the correct equation of the usual pendulum:

$$\ddot{\theta} + \Omega^2 \sin \theta = 0, \quad \text{where } \Omega \equiv \left(\frac{g}{R}\right)^{1/2}. \quad (2.26)$$

We will explore the full dynamic equation (25) in more detail later, but please note how simple its derivation was - in comparison with writing the Newton laws and then excluding the reaction force.

Next, though the Lagrangian formalism was derived from the Newton law for mechanical systems, the resulting equations (19) are applicable to other dynamic systems, especially those for which the kinetic and potential energies may be readily expressed via some generalized coordinates. As the simplest example, consider the well-known connection (Fig. 4) of a capacitor with capacitance  $C$  to an inductive coil with self-inductance  $L$ .<sup>10</sup> (Electrical engineers frequently call it the *LC tank circuit*.)

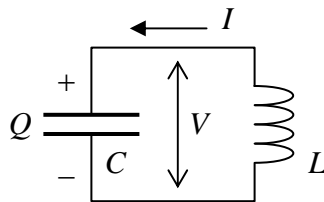


Fig. 2.4. *LC tank circuit*.

<sup>9</sup> This expression for  $T \equiv (m/2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$  may be readily obtained either by the formal differentiation of Eq. (2) over time, or just by noticing that the velocity vector has two perpendicular components: one along the ring (with magnitude  $R\dot{\theta}$ ) and another one normal to the ring plane (with magnitude  $\omega\rho = \omega R \sin \theta$  - see Fig. 1).

<sup>10</sup> Let me hope that this traditional notation would not lead to the confusion between the inductance and the Lagrange function.

As the reader certainly knows, at relatively low frequencies we may use the so-called lumped-circuit approximation, in which the total energy of the system as the sum of two components, the electric energy  $E_C$  localized inside the capacitor, and the magnetic energy  $E_L$  localized inside the inductance coil

$$E_C = \frac{Q^2}{2C}, \quad E_L = \frac{LI^2}{2}. \quad (2.27)$$

Since the electric current  $I$  through the coil and the electric charge  $Q$  on the capacitor are connected by the charge continuity equation  $dQ/dt = I$  (evident from Fig. 4), it is natural to declare the charge a generalized coordinate, and the current, the generalized velocity. With this choice, the electrostatic energy  $E_C(Q)$  should may be treated as the potential energy  $U$  of the system, and the magnetic energy  $E_L(I)$ , as its kinetic energy  $T$ . With this attribution, we get

$$\frac{\partial T}{\partial \dot{q}} \equiv \frac{\partial E_L}{\partial I} = LI \equiv L\dot{Q}, \quad \frac{\partial T}{\partial q} \equiv \frac{\partial E_L}{\partial Q} = 0, \quad \frac{\partial U}{\partial q} \equiv \frac{\partial E_C}{\partial Q} = \frac{Q}{C}, \quad (2.28)$$

so that the Lagrange equation of motion is

$$\frac{d}{dt}(L\dot{Q}) - \left(-\frac{Q}{C}\right) = 0. \quad (2.29)$$

Note, however, that the above choice of the generalized coordinate and velocity is not unique. Instead, one can use as the generalized coordinate the magnetic flux  $\Phi$  through the inductive coil, related to the common voltage  $V$  across the circuit (Fig. 4) by Faraday's induction law  $V = -d\Phi/dt$ . With this choice,  $(-V)$  becomes the generalized velocity,  $E_L = \Phi^2/2L$  should be understood as the *potential* energy, and  $E_C = CV^2/2$  treated as the *kinetic* energy. It is straightforward to verify that for this choice, the resulting Lagrange equation of motion is equivalent to Eq. (29). If both parameters of the circuit,  $L$  and  $C$ , are constant in time, Eq. (29) is just the harmonic oscillator equation similar to Eq. (1.1), and describes sinusoidal oscillations with frequency

$$\omega_0 = \frac{1}{(LC)^{1/2}}. \quad (2.30)$$

This is of course a very well known result that may be derived in the more standard way by equating the voltage drops across the capacitor ( $V = Q/C$ ) and the inductor ( $V = -LdI/dt = -Ld^2Q/dt^2$ ). However, the Lagrangian approach is much more convenient for more complex systems, for example, for the description of electromagnetic field and its interaction with charged relativistic particles.<sup>11</sup>

### 2.3. Hamiltonian function and energy

The canonical form (19) of the Lagrange equation has been derived using Eq. (18), which is formally similar to Eq. (1.22) for a potential force. Does this mean that the system described by Eq. (19) always conserves energy? Not necessarily, because the ‘‘potential energy’’  $U$ , that participates in Eq. (18), may depend not only on the generalized coordinates, but on time as well. Let us start the analysis of this issue with the introduction of two new (and very important!) notions: the *generalized momenta* corresponding to each generalized coordinate  $q_j$ ,

<sup>11</sup> See, e.g., EM Sec. 9.8.

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j}, \quad (2.31) \quad \text{Generalized momentum}$$

and the *Hamiltonian function*<sup>12</sup>

$$H \equiv \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L \equiv \sum_j p_j \dot{q}_j - L. \quad (2.32) \quad \text{Hamiltonian function}$$

In order to see whether the Hamiltonian function is conserved, let us differentiate its definition (32) over time:

$$\frac{dH}{dt} = \sum_j \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) \dot{q}_j + \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j \right] - \frac{dL}{dt}. \quad (2.33)$$

If we want to make use of the Lagrange equation (19), the last derivative has to be calculated considering  $L$  as a function of independent arguments  $q_j$ ,  $\dot{q}_j$ , and  $t$ :

$$\frac{dL}{dt} = \sum_j \left( \frac{\partial L}{\partial q_j} \dot{q}_j + \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j \right) + \frac{\partial L}{\partial t}, \quad (2.34)$$

where the last term is the derivative of  $L$  as an *explicit* function of time. We see that the last term in the square brackets of Eq. (33) immediately cancels with the last term in the parentheses of Eq. (34). Moreover, using the Lagrange equation (19) for the first term in the square brackets of Eq. (33), we see that it cancels with the first term in the parentheses of Eq. (34). Thus we arrive at a very simple and important result:

$$\frac{dH}{dt} = - \frac{\partial L}{\partial t}. \quad (2.35) \quad \text{Hamiltonian function's evolution}$$

The most important corollary of this formula is that if the Lagrangian function does not depend on time *explicitly* ( $\partial L / \partial t = 0$ ), the Hamiltonian function is an integral of motion:

$$H = \text{const.} \quad (2.36)$$

Let us see how it works, using the first two examples discussed in the previous section. For a 1D particle, definition (31) of the generalized momentum yields

$$p_x \equiv \frac{\partial L}{\partial v} = mv, \quad (2.37)$$

so that it coincides with the usual momentum - or rather with its  $x$ -component. According to Eq. (32), the Hamiltonian function for this case (with just one degree of freedom) is

$$H \equiv p\dot{x} - L = m\dot{x}^2 - \left( \frac{m}{2} \dot{x}^2 - U \right) = \frac{m}{2} \dot{x}^2 + U, \quad (2.38)$$

<sup>12</sup> It is sometimes called just the “Hamiltonian”, but it is advisable to use the full term “Hamiltonian function” in classical mechanics, in order to distinguish it from the *Hamiltonian operator* used in quantum mechanics. (Their relation will be discussed in Sec. 10.1.)



and coincides with particle's mechanical energy  $E = T + U$ . Since the Lagrangian does not depend on time explicitly, both  $H$  and  $E$  are conserved.

However, it is not always that simple! Indeed, let us return again to our testbed problem (Fig. 1). In this case, the generalized momentum corresponding to the generalized coordinate  $\theta$  is

$$p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}} = mR^2 \dot{\theta}, \quad (2.39)$$

and Eq. (32) yields:

$$\begin{aligned} H \equiv p_\theta \dot{\theta} - L &= mR^2 \dot{\theta}^2 - \left[ \frac{m}{2} R^2 (\dot{\theta}^2 + \omega^2 \sin^2 \theta) + mgR \cos \theta \right] + \text{const} \\ &= \frac{m}{2} R^2 (\dot{\theta}^2 - \omega^2 \sin^2 \theta) - mgR \cos \theta + \text{const}. \end{aligned} \quad (2.40)$$

This means that (as soon as  $\omega \neq 0$ ), the Hamiltonian function *differs* from the mechanical energy

$$E \equiv T + U = \frac{m}{2} R^2 (\dot{\theta}^2 + \omega^2 \sin^2 \theta) - mgR \cos \theta + \text{const}. \quad (2.41)$$

The difference,  $E - H = mR^2 \omega^2 \sin^2 \theta$  (besides an inconsequential constant), may change at bead's motion along the ring, so that although  $H$  is an integral of motion (since  $\partial L / \partial t = 0$ ), energy  $E$  is *not* conserved.

Let us find out when do these two functions,  $E$  and  $H$ , coincide. In mathematics, there is a notion of a *homogeneous function*  $f(x_1, x_2, \dots)$  of degree  $\lambda$ , defined in the following way: for an arbitrary constant  $a$ ,

$$f(ax_1, ax_2, \dots) = a^\lambda f(x_1, x_2, \dots). \quad (2.42)$$

Such functions obey the following *Euler theorem*:<sup>13</sup>

$$\sum_j \frac{\partial f}{\partial x_j} x_j = \lambda f, \quad (2.43)$$

that may be readily proven by differentiating both parts of Eq. (42) over  $a$  and then setting this parameter to the particular value  $a = 1$ . Now, consider the case when the kinetic energy is a quadratic form of all generalized velocities  $\dot{q}_j$ :

$$T = \sum_{j,j'} t_{jj'}(q_1, q_2, \dots, t) \dot{q}_j \dot{q}_{j'}, \quad (2.44)$$

with no other terms. It is evident that such  $T$  satisfies the definition of a homogeneous function of the velocities with  $\lambda = 2$ ,<sup>14</sup> so that the Euler theorem (43) gives

$$\sum_j \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j = 2T. \quad (2.45)$$

<sup>13</sup> This is just one of many theorems bearing the name of the mathematics genius L. Euler (1707-1783).

<sup>14</sup> Such functions are called *quadratic-homogeneous*.

But since  $U$  is independent of the generalized velocities,  $\partial L / \partial \dot{q}_j = \partial T / \partial \dot{q}_j$ , and the left-hand part of Eq. (45) is exactly the first term in the definition (32) of the Hamiltonian function, so that in this case

$$H = 2T - L = 2T - (T - U) = T + U = E. \quad (2.46)$$

So, for the kinetic energy of the type (44), for example a free particle with the kinetic energy considered as a function of its Cartesian velocities,

$$T = \frac{m}{2}(v_x^2 + v_y^2 + v_z^2), \quad (2.47)$$

the notions of the Hamiltonian function and mechanical energy are identical. (Indeed, some textbooks, very regrettably, do not distinguish these notions at all!) However, as we have seen from our bead-on-the-rotating-ring example, this is not always true. For that problem, the kinetic energy, in addition to the term proportional to  $\dot{\theta}^2$ , has another, velocity-independent term – see the first of Eqs. (23) – and hence is *not* a quadratic-homogeneous function of the angular velocity.

Thus, Eq. (36) expresses a new conservation law, generally different from that of the mechanical energy conservation.

#### 2.4. Other conservation laws

Looking at the Lagrange equation (19), we immediately see that if  $L \equiv T - U$  as a whole is independent of some generalized coordinate  $q_j$ ,  $\partial L / \partial q_j = 0$ ,<sup>15</sup> then the corresponding generalized momentum is an integral of motion:<sup>16</sup>

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j} = \text{const.} \quad (2.48)$$

For example, for a 1D particle with Lagrangian (21), momentum  $p_x$  is conserved if the potential energy is constant (the  $x$ -component of force is zero) – of course. As a less obvious example, let us consider a 2D motion of a particle in the field of central forces. If we use polar coordinates  $r$  and  $\varphi$  in the role of the generalized coordinates, the Lagrangian function,<sup>17</sup>

$$L \equiv T - U = \frac{m}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r), \quad (2.49)$$

is independent of  $\varphi$  and hence the corresponding generalized momentum,

$$p_\varphi \equiv \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}, \quad (2.50)$$

<sup>15</sup> Such coordinates are frequently called *cyclic*, because in some cases (like in the second example considered below) they represent periodic coordinates such as angles. However, this terminology is misleading, because some “cyclic” coordinates (e.g.,  $x$  in our first example) have nothing to do with rotation.

<sup>16</sup> This fact may be considered a particular case of a more general mathematical statement called the *Noether theorem* (named after its author, A. E. Nöther, sometimes called the “greatest woman mathematician ever lived”). For its discussion see, e.g., Sec. 13.7 in H. Goldstein *et al.*, *Classical Mechanics*, 3<sup>rd</sup> ed. Addison Wesley, 2002.

<sup>17</sup> Note that here  $\dot{r}^2$  is just the square of the scalar derivative  $\dot{r}$ , rather than the square of vector  $\dot{\mathbf{r}} = \mathbf{v}$ .

is conserved. This is just a particular (2D) case of the angular momentum conservation – see Eq. (1.24). Indeed, for the 2D motion within the  $[x, y]$  plane, the angular momentum vector,

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} = \begin{vmatrix} \mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z \\ x & y & z \\ m\dot{x} & m\dot{y} & m\dot{z} \end{vmatrix}, \quad (2.51)$$

has only one nonvanishing component, perpendicular to the motion plane:

$$L_z = x(m\dot{y}) - y(m\dot{x}). \quad (2.52)$$

Differentiating the well-known relations between the polar and Cartesian coordinates,

$$x = r \cos \varphi, \quad y = r \sin \varphi, \quad (2.53)$$

over time, and plugging the result into Eq. (52), we see that  $L_z = mr^2 \dot{\varphi} = p_\varphi$ .

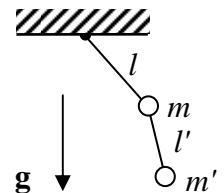
Thus the Lagrangian formalism provides a powerful way of searching for non-evident integrals of motion. On the other hand, if such conserved quantity is evident or known *a priori*, it is helpful for the selection of the most appropriate generalized coordinates, giving the simplest Lagrange equations. For example, in the last problem, if we have known in advance that  $p_\varphi$  had to be conserved, this could provide a motivation for using angle  $\varphi$  as one of generalized coordinates.

## 2.5. Exercise problems

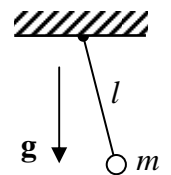
In each of Problems 2.1-2.10:

- (i) introduce a set of convenient generalized coordinate(s)  $q_j$  of the system,
- (ii) write down Lagrangian  $L$  as a function of  $q_j, \dot{q}_j$ , and (if appropriate) time,
- (iii) write down the Lagrangian equation(s) of motion,
- (iv) calculate the Hamiltonian function  $H$ ; find out whether it is conserved,
- (v) calculate energy  $E$ ; is  $E = H$ ?; is energy conserved?
- (vi) any other evident integrals of motion?

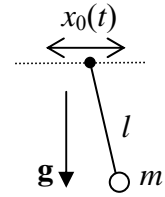
2.1. Double pendulum – see Fig. on the right. Consider only the motion confined to a vertical plane containing the suspension point.



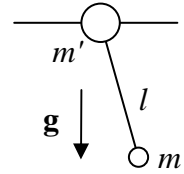
2.2. Stretchable pendulum (i.e. a mass hung on an elastic cord that exerts force  $F = -\kappa(l - l_0)$ , where  $\kappa$  and  $l_0$  are positive constants), confined to a vertical plane:



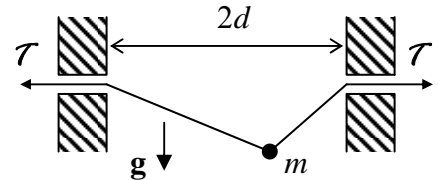
2.3. Fixed-length pendulum hanging from a horizontal support whose motion law  $x_0(t)$  is fixed. (No vertical plane constraint here.)



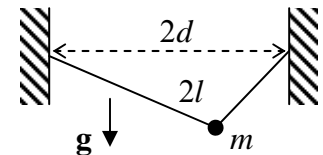
2.4. A pendulum of mass  $m$  hung on another point mass  $m'$  that may slide, without friction, along a straight horizontal rail (see Fig. on the right). Its motion is confined to the vertical plane that contains the rail.



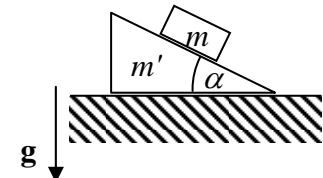
2.5. A bead of mass  $m$ , sliding without friction along a light string stretched by fixed force  $\mathcal{T}$ , between two horizontally displaced points – see Fig. on the right. Here, in contrast to the similar Problem 1.10, string tension  $\mathcal{T}$  may be comparable with bead's weight  $mg$ , and the motion is not restricted to the vertical plane.



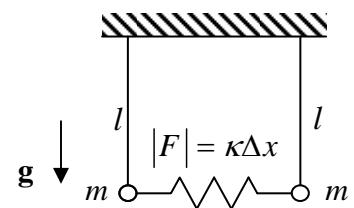
2.6. A bead of mass  $m$ , sliding without friction along a light string of fixed length  $2l$ , which is hung between two points, horizontally displaced by distance  $2d < 2l$  – see Fig. on the right. As in the previous problem, the motion is not restricted to the vertical plane.



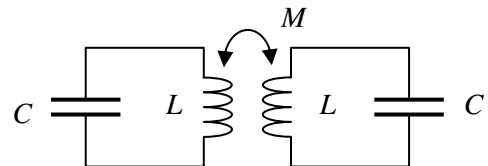
2.7. A block of mass  $m$  that can slide, without friction, along the inclined plane surface of a heavy wedge with mass  $m'$ . The wedge is free to move, also without friction, along a horizontal surface – see Fig. on the right. (Both motions are within the vertical plane containing the steepest slope line.)



2.8. The two-pendula system that was the subject of Problem 1.8 – see Fig. on the right.



2.9. A system of two similar, inductively-coupled LC circuits – see Fig. on the right.



2.10.\* A small Josephson junction, i.e. a system of two superconductors coupled by Cooper-pair tunneling through a thin insulating layer that separates them (see Fig. on the right).



*Hints:*

(i) At not very high frequencies (whose quantum  $\hbar\omega$  is lower than the binding energy  $2\Delta$  of the Cooper pairs), the Josephson effect may be described by coupling energy

$$U(\varphi) = -E_J \cos \varphi + \text{const},$$

where constant  $E_J$  describes the coupling strength, and variable  $\varphi$  (called the *Josephson phase difference*) is related to voltage  $V$  across the junction via the famous frequency-to-voltage relation

$$\frac{d\varphi}{dt} = \frac{2e}{\hbar} V,$$

where  $e \approx 1.6 \times 10^{-19}$  C is the fundamental electric charge and  $\hbar \approx 1.054 \times 10^{-34}$  J·s is the Plank constant.<sup>18</sup>

(ii) The junction (as any system of two close conductors) has a substantial electric capacitance  $C$ .

---

<sup>18</sup> More discussion of the Josephson effect and the physical sense of the variable  $\varphi$  may be found, for example, in EM Sec. 6.4 and QM Secs. 2.3 and 2.8 of this lecture note series.

This page is  
intentionally left  
blank